Reviving Quantum Geometrodynamics

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in collaboration with Thorsten Lang

arXiv:2305:09650, arXiv:2305.10097, arXiv:2311.00245
and forthcoming publications.

What is Quantum Geometrodynamics?

Classical Basics

- Earliest approach to the quantization of general relativity
  (DeWitt '67, Arnowitt et al. '62)
- Start from classical Hamiltonian formulation
- Canonical variables:
  Spatial metric $q_{ab}(x)$ and conjugate momentum $p^{ab}(x)$

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- Canonical variables:
  Spatial metric $q_{ab}(x)$ and conjugate momentum $p^{ab}(x)$
- First class system of Hamiltonian and diffeomorphism constraints:

$$\mathcal{H} = \frac{1}{\sqrt{q}}\left(q_{ac}q_{bd} - \frac{1}{n-1}q_{ab}q_{cd}\right)p^{ab}p^{cd} - \sqrt{q}R,$$

$$\mathcal{D}_a = -2D_b p^b{}_a$$

- Hamiltonian fully constrained

What is Quantum Geometrodynamics?

Quantization

- Naive canonical quantization:

$$\hat{q}_{ab}(x)\Psi[q_{ab}] = q_{ab}(x)\Psi[q_{ab}], \qquad \hat{p}^{ab}(x)\Psi[q_{ab}] = -\mathrm{i}\frac{\delta\Psi[q_{ab}]}{\delta q_{ab}(x)}$$

- Implementation of constraints in the quantum theory:

$$\mathcal{H}(\hat{q},\hat{p})\Psi = 0 \qquad \mathcal{D}_a(\hat{q},\hat{p})\Psi = 0$$

What is Quantum Geometrodynamics?

Open Questions

- How can we make sense of non–linear functions such as $\mathcal{H}(\hat{q}, \hat{p})$ of operator–valued distributions $\hat{q}_{ab}(x)$ and $\hat{p}^{ab}(x)$?

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Failure to address these and other issues led to the abandonment of quantum geometrodynamics
(Kiefer '07, Isham '91)

Other approaches

... and led to the birth of alternative approaches:

- Canonical LQG (Thiemann '07)
- Spin foams (Perez '03, Rovelli '07)
- Group field theory (Oriti '09)
- Causal dynamical triangulations (Loll '20)
- ...

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Common theme: Reformulate the theory and then adopt lattice regularizations in order to gain non–perturbative control

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Common theme: Reformulate the theory and then adopt lattice regularizations in order to gain non–perturbative control

A lattice regularization in the original ADM variables has never been tried!

Overview

1. Motivation ✓

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2. Forward Solutions
   2.1 A Regularization Scheme
   2.2 Quantum Theory with Positive–Def. Metric
3. Representation of Gauge Transformations
4. Continuum Limit
5. Summary and Outlook

2. Forward Solutions

2.1. A Regularization Scheme

2.1 Forward Solutions – A Regularization Scheme

General Idea

Regularization

- IR: Torus as spatial manifold
- UV: Restrict phase space of classical geometrodynamics to piecewise constant fields on a cubic lattice
- Replace derivatives ∂_a by finite differences Δ_a

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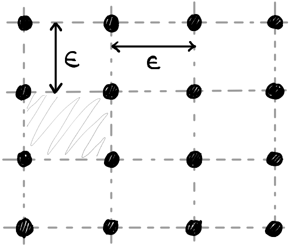
Implementation

- Evaluate constraints on restricted phase space
- Compute lattice corrections to constraint algebra
- Compute constraint algebra
- Quantize and study continuum limit

2.1 Forward Solutions – A Regularization Scheme

Example in two spatial dimensions



Restrict phase space of field variables $q_{ab}(x, y), p^{cd}(x, y)$ to piecewise constant fields, e.g.:

$$q_{ab}(x) = \sum_{X,Y=1}^{N} q_{ab}^{XY} \chi_{XY}(x)$$

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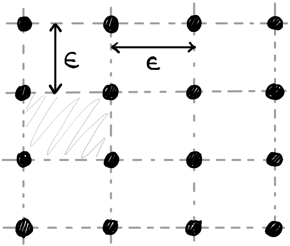

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$$q_{ab}(x) = \sum_{X,Y=1}^{N} q_{ab}^{XY} \chi_{XY}(x)$$

- Associate lattice degrees of freedom q_{ab}^{XY} to the lattice site (X, Y)
- Lattice degrees of freedom inherit Poisson bracket algebra from continuum fields:

$$\left\{ q_{ab}^{X_1 Y_1}, p_{X_2 Y_2}^{cd} \right\} = \frac{1}{\epsilon^2} \delta_a^{(c} \delta_b^{d)} \delta_{X_1}^{X_2} \delta_{Y_1}^{Y_2}$$

- Torus regularization implies periodic boundary conditions

2.1 Forward Solutions – A Regularization Scheme

Evaluation of the constraints on the restricted phase space yields lattice regularized constraints:

$$\mathcal{H}[N] = \epsilon^2 \sum_{XY} N^{XY} \left(\frac{1}{\sqrt{q}} \left(q_{ac} q_{bd} - \frac{1}{n-1} q_{ab} q_{cd} \right) p^{ab} p^{cd} - \sqrt{q} R \right)^{XY}$$

$$\mathcal{D}_a[N^a] = \epsilon^2 \sum_{XY} N^a_{XY} \left(-2\Delta_b (q_{ac} p^{cb}) + (\Delta_a q_{bc}) p^{bc} \right)^{XY}$$

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Note: Chain rule for finite differences acquires extra term proportional to lattice constant

$\Rightarrow$ necessity of a choice regarding the order of execution

2.1 Forward Solutions – A Regularization Scheme

Constraint algebra on the lattice acquires extra terms:

$$\left\{\mathcal{D}[\vec{M}],\mathcal{D}[\vec{N}]\right\} = \mathcal{D}[\mathcal{L}_{\vec{M}}\vec{N}] + \epsilon\, A_{\mathcal{D}\mathcal{D}}(\vec{M},\vec{N}),$$
$$\left\{\mathcal{D}(\vec{\mathcal{N}}),\mathcal{H}[N]\right\} = \mathcal{H}(\mathcal{L}_{\vec{N}}N) + \epsilon\, A_{\mathcal{D}\mathcal{H}}(\vec{N},N),$$
$$\{\mathcal{H}[M],\mathcal{H}[N]\} = \mathcal{D}[F(M,N)] + \epsilon\, A_{\mathcal{H}\mathcal{H}}(M,N)$$

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- Unphysical degrees of freedom
- Suppressed on fine lattices $\epsilon \to 0$

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Hint for continuum limit: Tune the limit such that long time evolutions are matched with sufficiently fine lattice spacings in order to control the deviation from the constraint surface

2. Forward Solutions

2.2. Quantum Theory with Pos.–Def. Metric

2.2 Forward Solutions – Quantum Theory

Standard Schrödinger Representation

$$(\hat{q}_{ab}^{XY}\psi)(q) = q_{ab}^{XY}\psi(q)$$
$$(\hat{p}_{XY}^{ab}\psi)(q) = -i\frac{\partial}{\partial q_{ab}^{XY}}\psi(q)$$

with $\psi(q) \in \mathcal{H} = L^2\left(\mathbb{R}^3, \mathrm{d}q_{ab}\right)$ for each lattice site (X, Y)

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with $\psi(q) \in \mathcal{H} = L^2\left(\mathbb{R}^3, \mathrm{d}q_{ab}\right)$ for each lattice site (X, Y)

Satisfy standard commutation relations

$$\left[\hat{q}^{X_1Y_1}_{ab}, \hat{p}^{cd}_{X_2Y_2}\right] = \frac{1}{\epsilon^2}\delta^{(c}_{a}\delta^{d)}_{b}\delta^{X_2}_{X_1}\delta^{Y_2}_{Y_1},$$
$$\left[\hat{q}^{X_1Y_1}_{ab}, \hat{q}^{X_2Y_2}_{cd}\right] = \left[\hat{p}^{ab}_{X_1Y_1}, \hat{p}^{cd}_{X_2Y_2}\right] = 0$$

States can have support on non–positive definite metrics – causal structure lost!

2.2 Forward Solutions – Quantum Theory

Our idea of using a different representation

- … that ensures positive–definiteness
- but keeps the standard commutation relations

Cholesky Decomposition

Every positive definite matrix q can be decomposed into the product

$$q = u^{\mathsf{T}} u,$$

where u is an upper triangular matrix with positive diagonal elements. This decomposition is unique.

Note that $\mathrm{UT}_{+}(2, \mathbb{R})$ is a Lie group.

2.2 Forward Solutions – Quantum Theory

Use this Lie group $\mathrm{UT}_+(2,\mathbb{R})$ to construct a Hilbert space:

$$\mathcal{H} = L^2(\mathrm{UT}_+(2,\mathbb{R}), \rho(u)\mathrm{d}u)$$

where $\rho(u)$ is the left Haar measure associated with $\mathrm{UT}_+(2,\mathbb{R})$

Representation of $\hat{q}_{ab}^{XY}$ on $\mathcal{H}$

$$\begin{aligned}
(\hat{q}_{11}\psi)(u) &= u_{11}^2\psi(u), \\
(\hat{q}_{12}\psi)(u) &= u_{11}u_{12}\psi(u), \\
(\hat{q}_{22}\psi)(u) &= \left(u_{12}^2 + u_{22}^2\right)\psi(u).
\end{aligned}$$

Realizes positive–definiteness of the spatial metric

How to represent the momentum operator?

2.2 Forward Solutions – Quantum Theory

First, define generators of shifts in positive q–directions

$$U(s_{ab})\hat{q}_{ab}U(s_{ab})^{\mathsf{T}} = \hat{q}_{ab} + s_{ab},$$

where $s_{ab} > 0$. The following $U(s)$ does the job

$$(U(s)\psi)(u) = \sqrt{\frac{\det J_q(u)}{\det J_q(g_s(u))}\frac{\rho(g_s(u))}{\rho(u)}}\psi(g_s(u)),$$

where g_s is a diffeo on $UT_+(2,\mathbb{R})$ with $g_s(u) = q^{-1}(q(u) + s)$.

One can show that $\left\{U(s) \in B(\mathcal{H}), s \in \mathbb{R}^3\right\}$ forms a strongly continuous contraction semigroup.

2.2 Forward Solutions – Quantum Theory

To define the momentum operators, we use that the contraction semigroup $\left\{U(s) \in B(\mathcal{H}), s \in \mathbb{R}^{n(n+1)/2}\right\}$ admits the infinitesimal generators

$$i\hat{p}^{ab}\,\psi = \left(\frac{\mathrm{d}}{\mathrm{d}s_{ab}} U(s)\psi\right)_{s=0}.$$

This yields

$$i\hat{p}^{11} = \frac{1}{2u_{11}}\frac{\partial}{\partial u_{11}} - \frac{u_{12}}{2u_{11}^2}\frac{\partial}{\partial u_{12}} + \frac{u_{12}^2}{2u_{11}^2 u_{22}}\frac{\partial}{\partial u_{22}} - \frac{2u_{22}^2 + u_{12}^2}{2u_{11}^2 u_{22}^2},$$
$$i\hat{p}^{12} = \frac{1}{u_{11}}\frac{\partial}{\partial u_{12}} - \frac{u_{12}}{u_{11} u_{22}}\frac{\partial}{\partial u_{22}} + \frac{u_{12}}{u_{11} u_{22}^2},$$
$$i\hat{p}^{22} = \frac{1}{2u_{22}}\frac{\partial}{\partial u_{22}} - \frac{1}{2u_{22}^2}.$$

2.2 Forward Solutions – Quantum Theory

With this representation, $\hat{q}_{ab}^{XY}$ and $\hat{p}_{XY}^{cd}$ satisfy the standard commutation relations

$$\left[\hat{q}_{ab}^{X_1 Y_1}, \hat{p}_{X_2 Y_2}^{cd}\right] = i\delta_a^{(c}\delta_b^{d)}\delta_{X_1}^{X_2}\delta_{Y_1}^{Y_2},$$
$$\left[\hat{q}_{ab}^{X_1 Y_1}, \hat{q}_{cd}^{X_2 Y_2}\right] = \left[\hat{p}_{X_1 Y_1}^{ab}, \hat{p}_{X_2 Y_2}^{cd}\right] = 0.$$

At the same time, $\hat{q}_{ab}^{XY}$ is positive definite in the sense that

$$\hat{q}_{ab}s^a s^b$$

is a positive operator for any s.

3. Representation of Gauge Transformations

3. Representation of Gauge Transformations

- Restrict to theories whose constraints form a Lie algebra (e.g., the diffeo constraints)
- For illustrative purposes consider a scalar field theory

Classical continuum theory

General form of continuum constraint:

$$D[N] = \int_{\mathbb{T}} \mathcal{D}(\phi(x), \partial\phi(x), \pi(x), \partial\pi(x)) f(x) \mathrm{d}x$$

Satisfies first class Poisson bracket algebra:

$$\{D[f], D[g]\} = D[F(f, \partial f, g, \partial g)]$$

3. Representation of Gauge Transformations

Classical lattice theory

Use lattice discretization $\phi_n(x) = \sum_{k=1}^{N_n} \phi_{nk} \chi_{X_k}(x)$. Lattice constraints are given by:

$$D_n[f_n] = \sum_{k=1}^{N_n} \mathcal{D}(\phi_{nk}, \Delta^n \phi_{nk}, \pi_{nk}, \Delta^n \pi_{nk}) f_{nk} \epsilon_n$$

Algebra on the lattice:

$$\{D_n[f_n], D_n[g_n]\} = D_n[F_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)] + \epsilon_n G_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)$$

3. Representation of Gauge Transformations

Solve Hamilton's equations of motion on the lattice:

$$\frac{\mathrm{d}\phi_n[g_n]}{\mathrm{d}s} = \{\phi_n[g_n], D_n[f_n]\}$$

Solution only depends on inital data for ϕ_{nk} if $D_n[f_n]$ is of first order in π_{nk}. The Hamiltonian flow $\varphi_s^{D_n[f_n]}$ can be interpreted as an approximate gauge transformation.

3. Representation of Gauge Transformations

Quantum theory

Define approximate gauge transformation in the quantum theory on the lattice:

$$\left(U\left(\varphi_s^{D_n[f_n]}\right)\psi_n\right)((\phi_{nk})_k) = \sqrt{\det\left(J_{\varphi_s^{D_n[f_n]}}((\phi_{nk})_k)\right)}\,\psi_n(\varphi_s^{D_n[f_n]}((\phi_{nk})_k))$$

Forms a unitary one–parameter group $\Rightarrow$ generator exists

See Thiemann '22 for related approach

4. Continuum Limit

4. Continuum Limit

The Weyl algebra on the lattice is spanned by the exponentiated canonical variables:

$$W_n = \overline{\mathrm{span}\{e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]}\}}$$

Let $W = \varprojlim W_n$ be the inverse limit with identifications

$$\hat{\phi}_{n+1,2k} f_{n+1,2k} + \hat{\phi}_{n+1,2k+1} f_{n+1,2k+1} \equiv \hat{\phi}_{nk}(f_{n+1,2k} + f_{n+1,2k+1})$$

Choose a sequence ψ_n of states on every lattice. Define

$$\omega_n \left(e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) := \left\langle \psi_n, e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \psi_n \right\rangle .$$

If ω_n forms Cauchy sequence, define

$$\omega \left(\lim_{n\to\infty} e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) := \lim_{n\to\infty} \omega_n \left(e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) .$$

Use GNS–construction to obtain continuum Hilbert space.

5. Summary and Outlook

Summary

- Lattice regularized version of quantum geometrodynamics
- Non–standard representation of the canonical commutation relations with inherently positive definite metric
- Representation of approximate gauge transformations on the lattice
- Criterion for existence of continuum limit

Outlook

- Explore converging sequences of lattice states
- Study continuum limit of approximate gauge transformations
- Goal: Find a strongly continuous representation of the diffeomorphism group
- Use generalized Weyl transformation to represent lattice Hamiltonian constraints
- Study continuum limit (probably involves renormalization techniques)

Thank you for your attention!