Strong Gravity and the BKL Conjecture

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The Conjecture (Belinskii, Khalatnikov, Lifshitz) (1971)

Near to a singularity spatially separated points decouple, and the role of most forms of matter is negligible.

Problems

- Does not appear geometric
- Mathematical implications vague
- Only conjecture - no analytic proof.

Motivation for applying the BKL conjecture

- The Einstein field equations are hard to solve:
  - Similar to homogenous or isotropic cosmology - attempt to solve the EFEs in a simple case
- Numerical evidence supports the BKL conjecture:
  - Recently a lot of numerical work has been done (Garfinkle, Uggla, Elst, Ellis, Wainwright, Curtis, Moncrief, Berger...) providing support for the BKL conjecture.

Why Ashtekar Variables

Why Use Ashtekar Variables?

- Simple formulation
- Basic Variables $\widetilde{E}_i^a$, A_a^i - no inverses.
  - ADM Variables need q^{ab} and q_{ab}. Not well defined at singularity.
- Candidate for quantum theory
- Singularity Resolution

Our work is done in self dual variables - re-evaluation in real variables to come later.

Singularity Resolution

- Aim is to resolve a large class of cosmological singularities.
- Similar position to first finding singularities
  - First specific singularities found
  - Disagreement about how generic singularities are
  - Singularity proven

Ashtekar Variables

A brief recap of Ashtekar variables with units

We work with a densitized triad $\widetilde{E}$ and connection A. Early letters (a,b,c..) denote spatial indices, later (i,j,k...) internal.

$$\widetilde{E}^a_i \widetilde{E}^{b\,i} = \sqrt{q} q^{ab} \quad (1)$$

$$A_a{}^i = \frac{1}{G}(\Gamma_a{}^i - iK_a{}^i) \quad (2)$$

$$\widetilde{E}^a_i K_b{}^i = qK^a{}_b \quad (3)$$

$$\{A_a{}^i(x), \widetilde{E}^b_j(y)\} = i\delta^b_a \delta^i_j \delta^3(x-y) \quad (4)$$

$$\widetilde{E}^a_i \sim q_{ab} \sim 1 \quad (5)$$

$$A_a{}^i \sim \frac{M}{LT} \quad (6)$$

The Kasner Singularity

Bianchi I Solution

The Kasner space-time takes the form:

$$ds^2 = -d\tau^2 + \tau^{2p_1}dx^2 + \tau^{2p_2}dy^2 + \tau^{2p_3}dz^2 \quad (7)$$

$$p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2 \quad (8)$$

$$p_i \in \mathbb{R} \quad (9)$$

- One degree of freedom
- Homogeneous vacuum solution
- Unstable in perturbations
- Adding scalar field (stiff fluid) gives rise to a stable sector

Strong Gravity as a Toy Model

The Strong Coupling Limit

We examine the limit $G \to \infty$ whilst keeping $\widetilde{E}_i^a$ and A_b^j fixed.

Relationship with BKL Conjecture

Consider the derivative operator

$$D_a h_i = \partial_a h_i + \frac{G}{c} \epsilon_{ij}^{k} A_a^j h_k \qquad (10)$$

- From this $\partial_a \to 0$ is equivalent to $G \to \infty$.
- The strong coupling limit forms a toy model of the BKL conjecture - we ignore spatial derivatives.
- Not the entire picture - generally we see spatial derivatives suppressed rather than ignored.

Constraints in Strong Gravity

Full Constraints

$$\tilde{\tilde{S}} = 2c\, \tilde{E}_i^a \tilde{E}_j^b \partial_{[a} A_{b]k} \epsilon^{ijk} + 2G \tilde{E}_i^a \tilde{E}_j^b A_a{}^{[i} A_b{}^{j]} \quad (11)$$

$$V_a = c\, \tilde{E}_i^b (2\partial_{[a} A_{b]i} + G A_a{}^j A_b{}^k \epsilon^i{}_{jk}) \quad (12)$$

$$G_i = c\, \partial_a \tilde{E}_i^a + G \epsilon_{ijk} A_a{}^j \tilde{E}^{ak} \quad (13)$$

Reduced Constraints

$$\tilde{\tilde{S}} = 2G\, \tilde{E}_i^a \tilde{E}_j^b A_a{}^{[i} A_b{}^{j]} \quad (14)$$

$$V_a = G\, \tilde{E}_i^b A_a{}^j A_b{}^k \epsilon^i{}_{jk} \quad (15)$$

$$G_i = G\, \epsilon_{ijk} A_a{}^j \tilde{E}^{ak} \quad (16)$$

$$\{\tilde{\tilde{S}}, V_a\} \cong 0 \qquad \{\tilde{\tilde{S}}, G_i\} \cong 0 \qquad \{G_i, V_a\} \cong 0 \quad (17)$$

Simpler Variables

New Variable $\widetilde{M}_i^j$

We can introduce a new variable to simplify things: $\widetilde{M}_i{}^j = \widetilde{E}_i^a A_a{}^j$

$$\{\widetilde{M}_i^j(x), \widetilde{M}_k^l(y)\} = i(\widetilde{M}_i^l \delta_k^j + \widetilde{M}_k^j \delta_i^l)\delta^3(x-y) \quad (18)$$

$$\{\widetilde{E}_k^a(x), \widetilde{M}_i^j(y)\} = i\widetilde{E}_i^a \delta_k^j \delta^3(x-y) \quad (19)$$

Constraints in terms of $\widetilde{M}_i^j$

$$\widetilde{\widetilde{S}} = G((\widetilde{M}_i{}^i)^2 - \widetilde{M}_i{}^j \widetilde{M}_j{}^i) \quad (20)$$

$$= G(Tr(\widetilde{M})^2 - Tr(\widetilde{M}^2)) \quad (21)$$

$$G_i = G\epsilon_{ijk} M^{kj} \quad (22)$$

Equations of Motion

Time Evolution of $\widetilde{M}_i^j$ and $\widetilde{E}_i^a$

$$\dot{\widetilde{E}}^a{}_i = -i\underset{\sim}{N}(\widetilde{M}_k{}^k\delta_i{}^j - \widetilde{M}_i{}^j)\widetilde{E}_j^a \tag{23}$$

$$\dot{\widetilde{M}}_i{}^j = 0 \tag{24}$$

Commutativity of Strong Limit and Equations of Motion

$$\begin{array}{ccc} \text{Full Constraint} & \xrightarrow{\text{Large G limit}} & \text{Reduced Constraint} \\ \downarrow \text{Equation of Motion} & & \downarrow \text{Equation of Motion} \\ \text{Full Equation of Motion} & \xrightarrow{\text{Large G limit}} & \text{Reduced Equation of Motion} \end{array}$$

Equations of Motion

Reality Conditions

$$q \in \mathbb{R} \to \dot{q} \in \mathbb{R} \to -i\widetilde{M} \in \mathbb{R} \tag{25}$$

$-iM$ is a symmetric (from Gauss Constraint) real matrix and is therefore diagonalizable in some basis of our internal space.

Solutions

We can solve our simple diagonal equations in the lapse $N = 1$ case:

$$ds^2 = -d\tau^2 + \tau^{-2f_1}({}^oE^1{}_1)^2 dx^2 + \tau^{-2f_2}({}^oE^2{}_2)^2 dy^2 + \tau^{-2f_3}({}^oE^3{}_3)^2 dz^2 \tag{26}$$

With scalar constraint implying

$$\sum f_i = \sum f_i^2 = 1 \tag{27}$$

A New Calculus

The BKL Limit

We will examine the following scenario:

- $\widetilde{E}_i^a \partial_a \mathbf{X} \to 0$ where $\sqrt{Det(q)}T$ has limit $\forall T \in \mathbf{X}$
- $A_a^i, \widetilde{E}_i^a \in \mathbf{X}$
- $Det(q) \to 0$
- Unreasonable Assumption: $\Gamma_a^i = 0$

Why Use This Model?

- Similar to "almost FL" models of Uggla, Elst et al.
- Numerical evidence from Garfinkle
- Gowdy model: $\partial_a X$ suppressed
- Allows for $\frac{\partial_a f}{f}$ bounded.
- From Kasner model we see $\widetilde{E}_i^a, M_i^j$ have this behaviour.

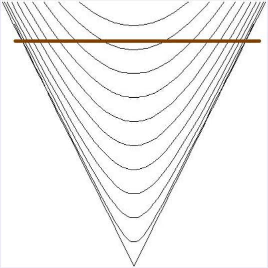
An Example

A non-singular example

Consider Minkowski space in coordinates adapted to a hyperboloid:

$$ds^2 = -d\rho^2 + \rho^2[d\chi^2 + cosh^2(\chi)(d\theta^2 + cos^2(\theta)d\phi^2)] \quad (28)$$

Here we see $\rho \to 0$ as taking $Det(q) \to 0$



Non-Reduced Constraints and Equations of Motion

Constraints

- Full constraints the same as in the toy model
- Multiple of the Gauss constraint added to Hamiltonian - keeps evolution real:

$$H = \int d^3x [\frac{1}{2} \underset{\sim}{N} \tilde{\tilde{S}} - iN^a V_a + (\tilde{E}^{ai} D_a \underset{\sim}{N}) G_i] \quad (29)$$

Equations of Motion

$$\dot{\tilde{E}}_i^b = -i\underset{\sim}{N} \tilde{E}_j^a D_a \tilde{E}_k^b \epsilon_i^{jk} \quad (30)$$

$$\dot{\tilde{M}}_i^j = 2i\underset{\sim}{N} \tilde{E}_i^b \tilde{E}_k^a \partial_{[a} A_{b]l} \epsilon^{jkl} - i\underset{\sim}{N} \tilde{E}_k^a \partial_a \tilde{E}_l^b \epsilon_i^{kl} A_b^j \quad (31)$$

$$-i\tilde{E}_i^b \partial_b (D_a \underset{\sim}{N} \tilde{E}^{aj}) - i\tilde{E}_i^b \epsilon_{kl}^j A_b^k D_a \underset{\sim}{N} \tilde{E}^{al} \quad (32)$$

$$+i\underset{\sim}{N} \tilde{M}^{kj} \tilde{M}_{[ik]} \quad (33)$$

Reduced Equations of Motion

Equations of Motion

$$G_i = Re[i\epsilon_{ijk} M^{jk}] \tag{34}$$

$$\dot{\tilde{E}}^a{}_i = -i\underset{\sim}{N}(\tilde{M}_k{}^k \delta_i{}^j - \tilde{M}_i{}^j)\tilde{E}^a_j \tag{35}$$

$$\dot{\tilde{M}}_i{}^j = 0 \tag{36}$$

Commutativity of BKL Reduction and Equations of Motion

$$\begin{array}{ccc} \text{Full Constraint} & \xrightarrow{\text{BKL Reduction}} & \text{Reduced Constraint} \\ \Big\downarrow \text{Equation of Motion} & & \Big\downarrow \text{Equation of Motion} \\ \text{Full Equation of Motion} & \xrightarrow{\text{BKL Reduction}} & \text{Reduced Equation of Motion} \end{array}$$

Reintroducing Spatial Curvature

Relaxing $\Gamma = 0$

- We can relax our $\Gamma = 0$ condition
- $i\widetilde{M}$ is no longer a symmetric matrix
- $\dot{\widetilde{M}}{}_i^j = i \underset{\sim}{N} \widetilde{M}^{kj} \widetilde{M}_{[ik]}$
- General analytic solutions not found
- 'Bounces' appear in numerical simulation
- Perturbations to symmetric matrix grow

Conclusions

Results

- BKL Conjecture can be applied in Ashtekar Variables
- Resulting spacetime has Kasner singularity in 'worst case'
- Corresponds to known numerical results

Open Issues

- Work done in self-dual variables - changes in real variables?
- Spatial curvature is non-zero in general for bounce behaviour
- Need to add matter

References

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