Localized energy density of GWs

Simone Speziale

ILGQs, 5-4-24

based on work with:

Abhay Ashtekar

Antoine Rignon-Bret

Null Infinity as a Weakly Isolated Horizon
2402.17977

Localised energy of GWs
to appear

Centre de Physique Théorique

Apology...

Today I will not talk about loop quantum gravity

However, my motivation for the current work originates from it

PHYSICAL REVIEW D **91,** 064043 (2015)

First order gravity on the light front

Sergei Alexandrov[1] and Simone Speziale[2]

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(Received 14 January 2015; published 19 March 2015)

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Null twisted geometries

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(which can be visualized as `fuzzy twisted geometries')
when canonical quantization is carried out on a ***null*** hypersurface,
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when canonical quantization is carried out on a ***null*** hypersurface,
where constraint-free data are accessible? (Sachs, Rendall, …)

…this question sent me down a rabbit hole full of wonders…

… and great collaborators, Zhang, Alexandrov, Wieland, Ashtekar, De Paoli, Oliveri, Odak, Freidel, Pranzetti, Rignon-Bret…

Overview

Today's talk will focus on one of the most classics and at the same time deep questions of GR:
how do we define gravitational energy?
do gravitational waves carry energy?

1. Introduction to the problem (broad)
2. Energy of gravitational waves at future null infinity (work with Abhay)
3. Localized energy of gravitational waves at future null infinity (work with Antoine)

I - Hamiltonians generators and radiation

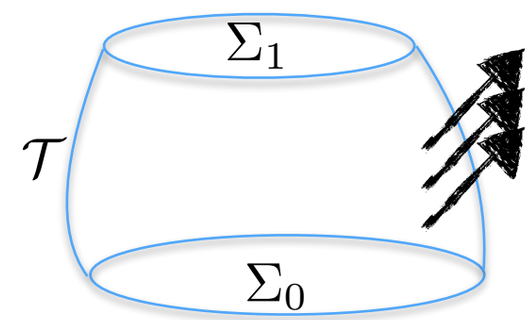
Energy and boundary conditions

In classical mechanics, conservative systems admit an elegant definition of energy as the Hamiltonian generator of time translations

$$i_{\partial_t}\omega = dH \qquad\qquad H = K + V$$

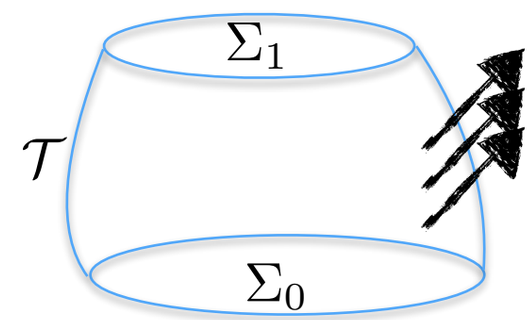
In field theory, conservation may depend on boundary conditions

e.g. a relativistic scalar field has wave solutions that can exit/enter a finite region thus making the system dissipative



Energy and boundary conditions

What can we say in the presence of dissipation caused for instance by outgoing waves?



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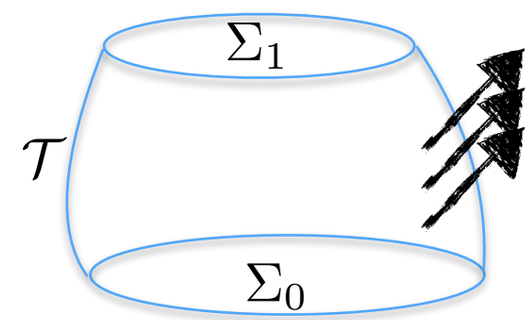
What can we say in the presence of dissipation caused for instance by outgoing waves?

The symplectic 2-form is closed on-shell: $d\omega \approx 0$

Hence by Stokes theorem: $$\int_{\Sigma_0} \omega = \int_{\Sigma_1} \omega + \int_{\mathcal{T}} \omega$$



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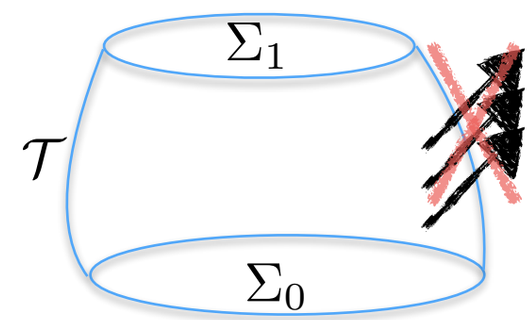

Typical set-up for conservation is to send the time-like boundary to infinity, with fall-off conditions so that the fields vanish

Then

$$I_{\partial_\tau}\Omega_\Sigma = \delta H \qquad H = \int_\Sigma \mathcal{H}, \qquad \mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \dots$$

local energy density

Notation:

$$d, i_X, \pounds_X, \qquad \delta, I_X, \delta_X$$

Energy and boundary conditions

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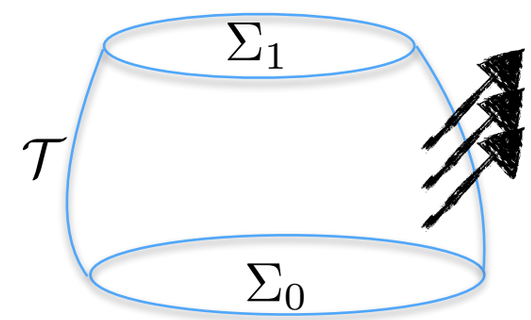


In the presence of outgoing or incoming flux, the symplectic form is *not conserved in time*

This situation makes also the notion of HVF tricky:

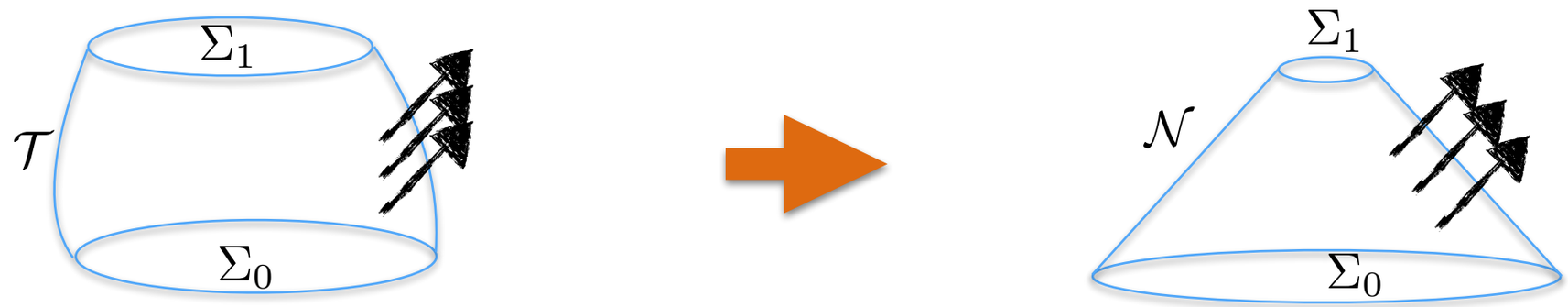
there are non-HVF which carry useful physical information!

e.g. symmetries producing normal deformations of the corner S

The presence of non-HVFs is nothing deep: if the system is dissipative, one can not expect vector fields whose motion is sensitive to the dissipation to be Hamiltonian

Different phase spaces

To make the description simpler, let us assume that the boundary is null



No mixing of outgoing/incoming flux, only outgoing one

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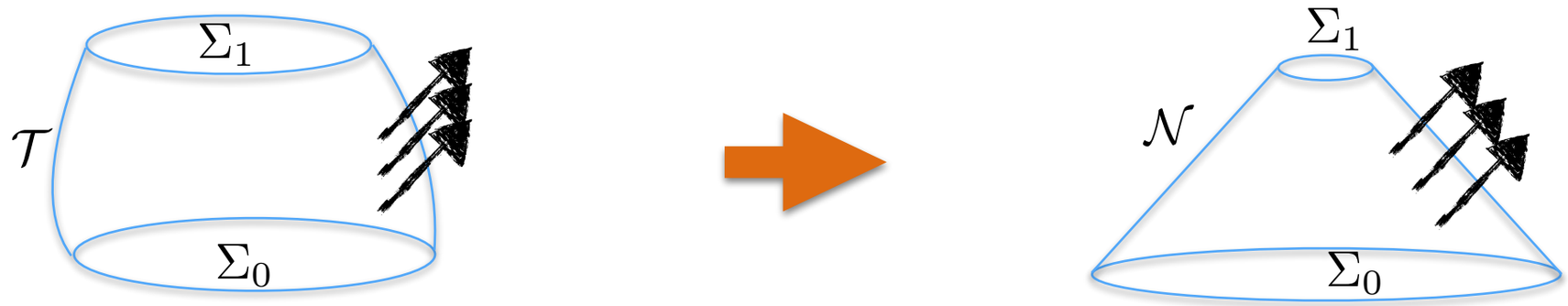
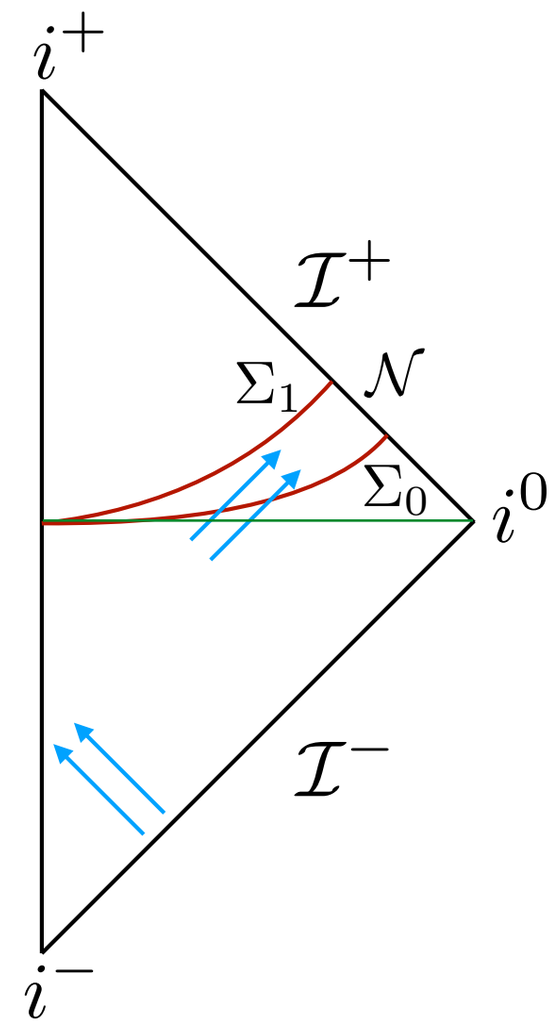


No mixing of outgoing/incoming flux, only outgoing one

If *N* is a region of *future null infinity*, then Σ are partial Cauchy slices

- Ω_{Σ_0} contains the Cauchy data on Σ_0
- $\Omega_{\mathcal{N}}$ contains the *radiative* data on $\mathcal{N}$
- Ω_{Σ_1} contains the Cauchy data on $\Sigma_0 \cup \mathcal{N}$



Obtained integrating the current $\Omega_\Sigma = \int_\Sigma \omega \qquad \Omega_{\mathcal{N}} = \int_{\mathcal{N}} \omega \qquad \omega = \delta\theta$

Different Hamiltonian generators

Ashtekar-Streubel '81, Iyer-Wald '94, Wald-Zoupas '99

In general, for a symmetry ξ we have

$$I_\xi \omega = \delta j_\xi - d i_\xi \theta \qquad \text{where} \qquad j_\xi = I_\xi \theta - i_\xi L \qquad \text{is the Noether current}$$

e.g. for a scalar field in Minkowski, ξ is a Killing vector, and j_ξ the e-m tensor

- $\int_{\Sigma_0} j_\xi$ is the total energy-momentum on Σ_0
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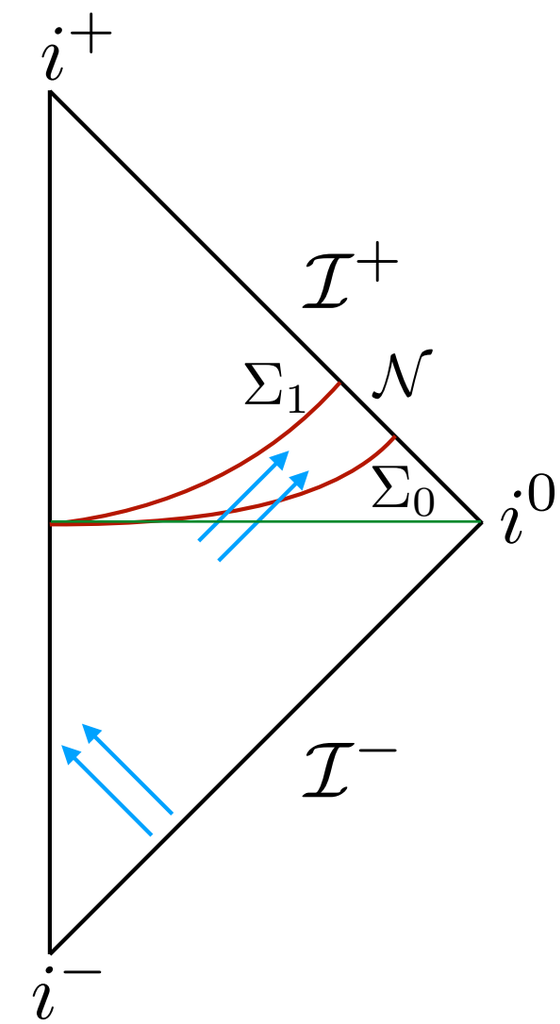

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If we can discard the `non-integrable' term $i_\xi \theta$
(namely a non exact 1-form in field space)
then these quantities are the Hamiltonians, and the symmetry a HVF

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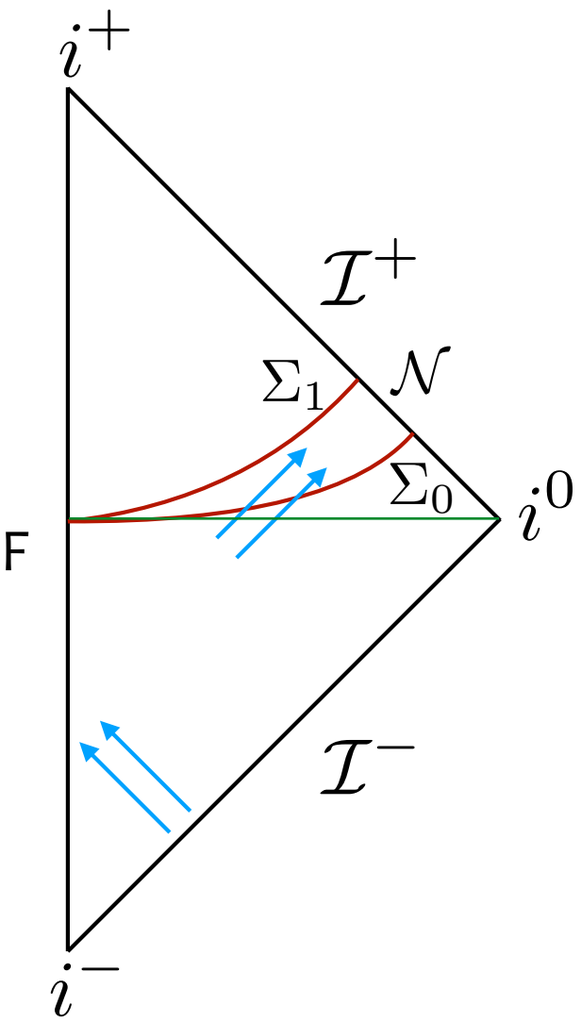

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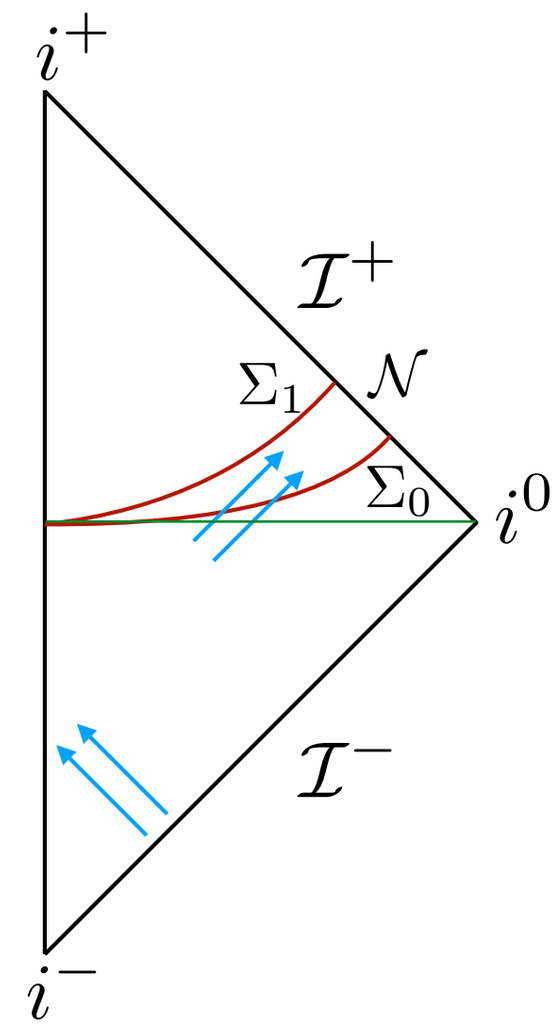
here $\quad i_\xi \theta = \pi \delta\phi \, i_\xi \epsilon_{\mathcal{N}}$

Examples of integrability:

- vanishes for ξ tangent to the corner
- vanishes if π=0 or δφ=0 (stationary/cons. b.c.)



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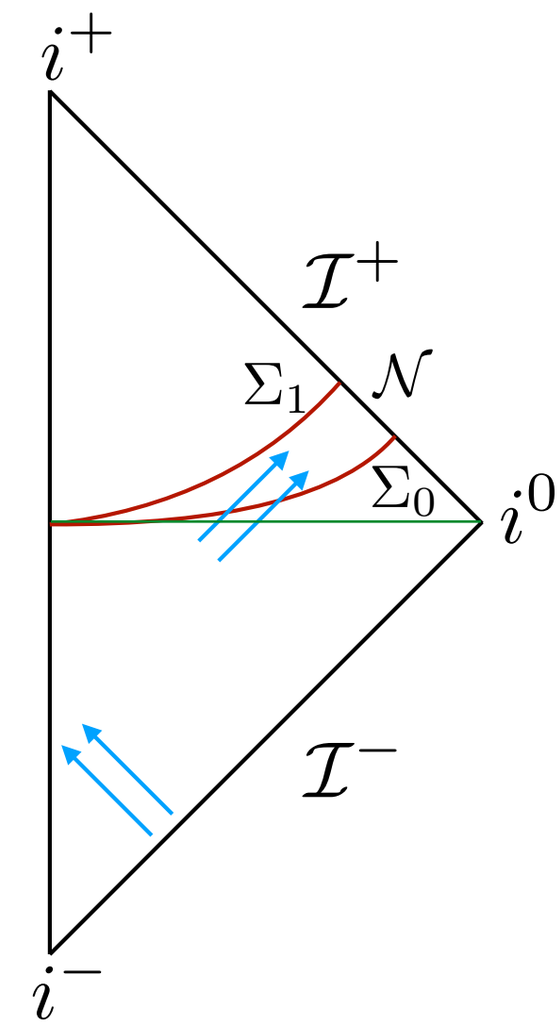
Examples of integrability:

- vanishes for ξ tangent to the corner
- vanishes if π=0

notice that this is a `dense' subset

We can make this intuition rigorous if we define a topology on the radiative phase space induced by a ``momentum''-norm

⇨ **obtain the generator on the full phase space by continuity**



Integrability via topology

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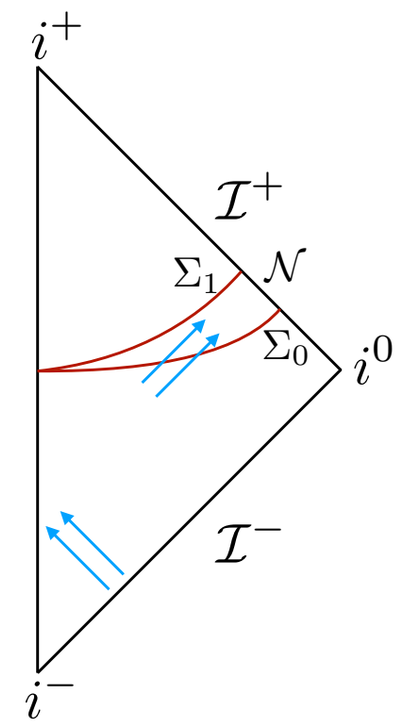
This integrability discussion is valid beyond the scalar field, and quite general



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The Hamiltonian exists if $\quad i_\xi \theta = \delta B$

Examples: ξ tangent to the cross-sections; $\quad \vartheta$ vanishes

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Hamiltonian in a weak sense:
the charges are generators
only for perturbations around
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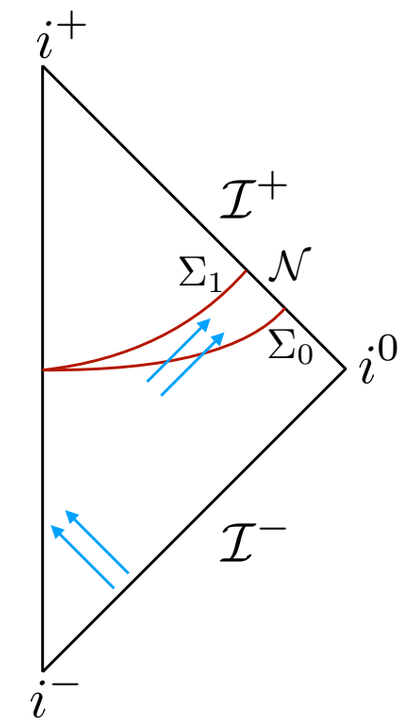

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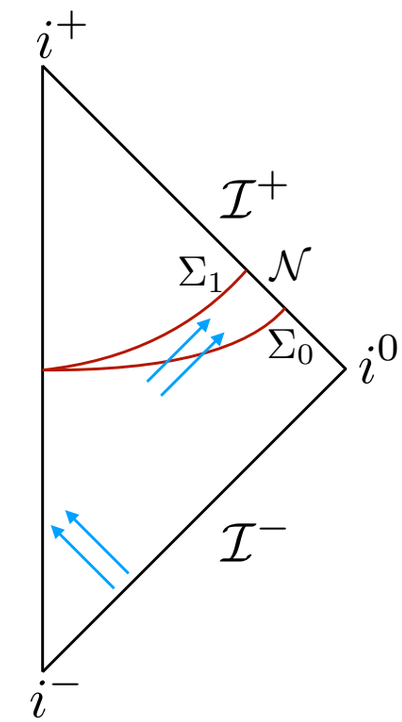

More general procedure at N:

- endow the radiative phase space with a topology;
- if this is such that $\theta = 0$ on a dense subset,
- then extend by continuity to define the Hamiltonian generator everywhere in the phase

In the case studied, it reproduces the same results as the Wald-Zoupas procedure

⇨ in the presence of radiations, surface charges at $\partial\Sigma$ provide Hamiltonians only in a weak sense whereas their fluxes across *N* provide Hamiltonians for the full radiative phase space

II - General relativity

Gravitational energy

General covariance prevents the definition of a local gravitational energy density

The intuitive reason is that by the equivalence principle we can locally set to zero the gravitational field (metric locally flat with zero first derivatives),
hence any candidate energy momentum $(\partial g)^2$ would vanish,
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- Use $(\partial^2 g)^2$: Belinfante tensor
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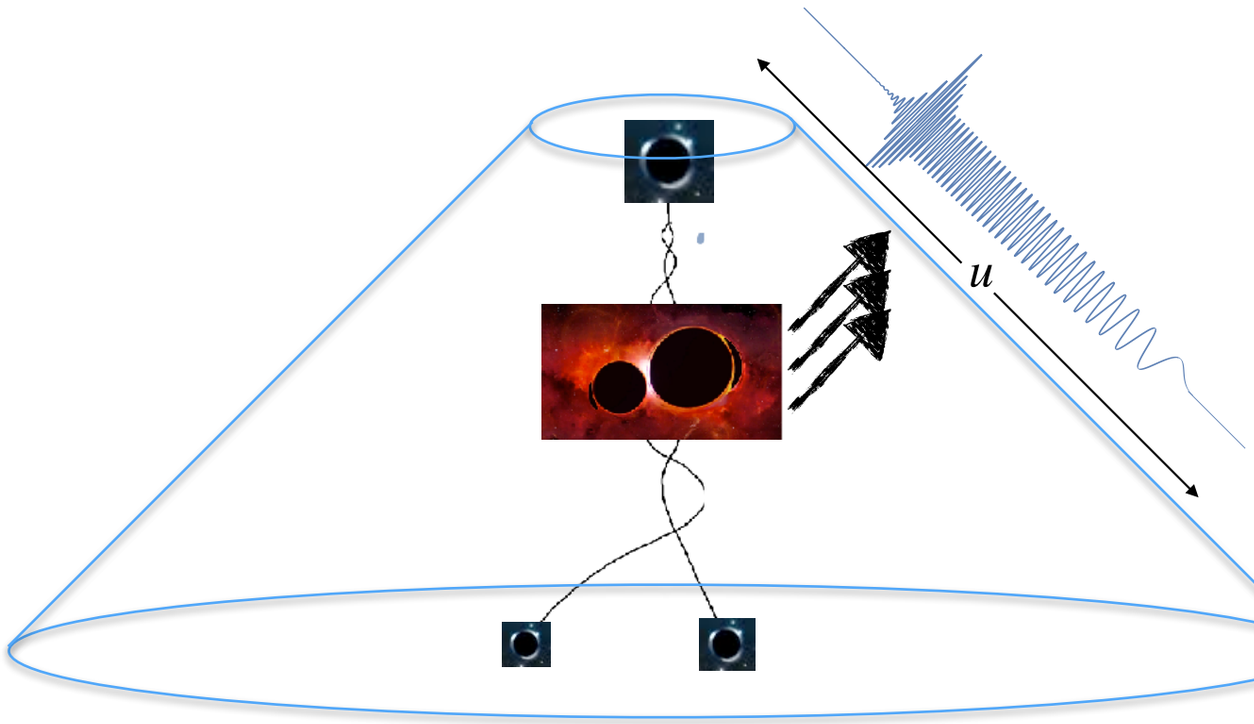
here the problem is solved by a class of `quasi-local'observables: surface integrals

Bondi energy-loss formula

The question for GWs was considered settled by Bondi's energy-loss formula:
a coordinate-independent statement on the amount of energy carried away by GWs



$$E(u_1) - E(u_0) = -\frac{1}{8}\int_{u_0}^{u_1}\int_{S^2} N_{ab}N^{ab}$$

Considered historically as `theoretical proof' that GWs exist,
in the sense that they can carry energy,
it is the first of a series of *flux-balance laws* that allow
the reconstruction and physical interpretation of a GW signal

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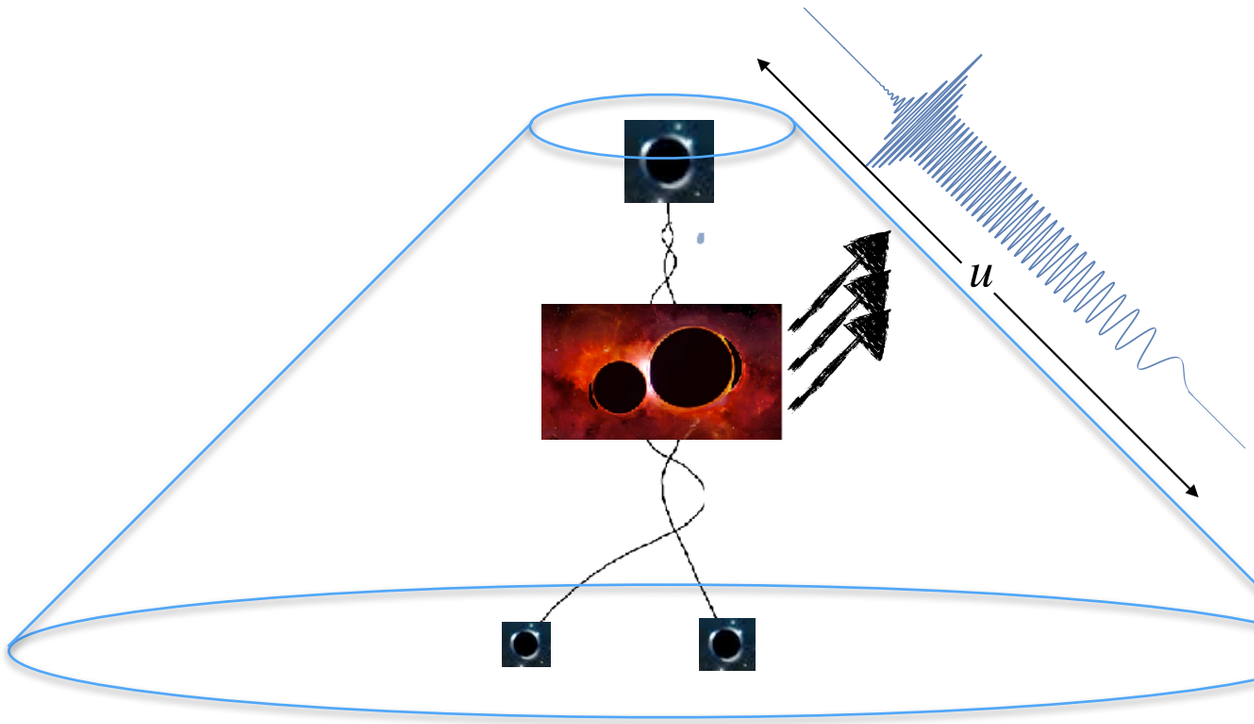

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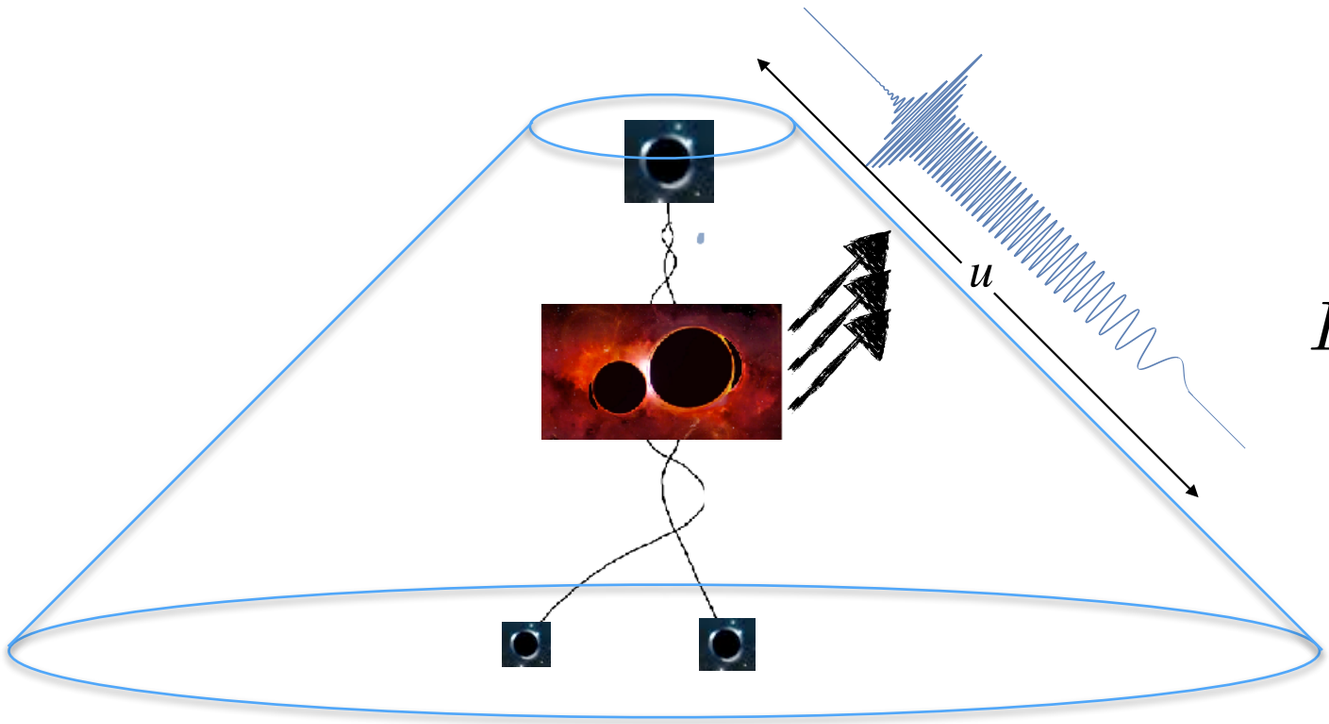

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But what do we even mean by the energy E here?

It is a surface integral, hence a notion of `global energy'

- The LHS and RHS can be *uniquely* identified as Hamiltonians
- This viewpoint offers a starting ground for a rich series of recent developments, ranging from the phenomenology of GW to theoretical GR to more generally the role of boundary symmetries in gauge theories and gravity

Surface charges

$$j_\xi = I_\xi \theta - i_\xi L$$

$$I_\xi \omega = \delta j_\xi - d i_\xi \theta$$

Surface charges

In gravity (and gauge theories) the Noether current is **exact** on-shell,

$$j_\xi = I_\xi \theta - i_\xi L \ = dq_\xi$$

surface charge

Ex. $$j^\mu_\xi = -\frac{1}{8\pi G}\nabla_\nu \nabla^{[\mu}\xi^{\nu]}$$

$$j^\mu_\lambda = \partial_\nu(\lambda F^{\mu\nu})$$

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- bulk diffeomorphisms (or gauge transf) are degenerate directions
  → **pure gauge**

- boundary diffeomorphisms (or gauge transf) are not:
  → **possible symmetries** (depends on boundary conditions and what diffeos are allowed)

Example of spatial infinity: under the ADM fall-off conditions, there are left-over diffeos compatible with the b.c. at i^0 that span the Poincare group

The `improved generator' of Regge-Teitelboim is really just this Hamiltonian generator

$$H := \int_\Sigma \mathcal{H} + E_{\rm ADM} \approx E_{\rm ADM}$$

Identifying fluxes

$$I_\xi \omega = \delta j_\xi - d i_\xi \theta \; = d(\delta q_\xi - i_\xi \theta)$$

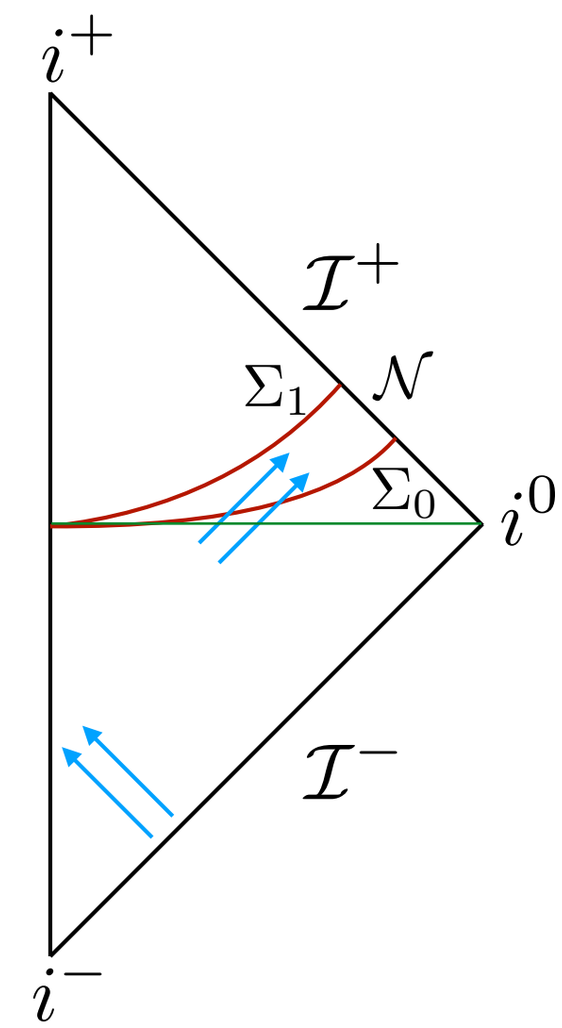
- The ADM case study is particularly simple, because in this case the fall-off conditions imply that

$\theta = \delta \ell$ ⇨ integrability follows trivially,
and the Hamiltonian generator is just a shift from the Komar formulas

this also means that $\Omega_{\mathcal{T}_\infty} = 0$ namely the system is conservative

- But in the limit to Scri instead, $\theta = \bar{\theta} + \delta \ell$ and $\Omega_{\mathcal{N}} \neq 0$

there is flux and the system is dissipative



Ideally, we would like to identify the non-radiative subset
of the phase space as the set of solutions for which $\bar{\theta} = 0$

Problem:

$\bar{\theta}$ is ambiguous; I can always move terms from ℓ to $\bar{\theta}$,
and this changes which solutions are identified as `stationary' !

Identifying fluxes

$$I_\xi \omega = \delta j_\xi - d i_\xi \theta \; = d(\delta q_\xi - i_\xi \theta)$$

freedom: $\bar{\theta} = \theta + \delta \ell$

$\bar{q}_\xi = q_\xi + i_\xi \ell$

Two alternative procedures:

- Wald-Zoupas: narrow down a preferred $\bar{\theta}$ with a list of requirements

- Ashtekar-S: the choice of topology
  that makes the Hamiltonian integrable in the radiative phase space
  also selects a preferred $\bar{\theta}$

In the examples studied, the results match (BMS, NEH, general null hypersurfaces)

BMS fluxes

When applied to the BMS symmetries, we obtain the Ashtekar-Streubel flux

$$F_\xi^{\mathrm{BMS}} = -\frac{1}{16\pi G}\int_{S_1}^{S_2} N_{ab}\delta_\xi \sigma^{ab}\epsilon_{\mathscr{I}}$$

$$\delta_\xi \sigma_{ab} = (\pounds_\xi - \dot{f})\sigma_{ab} + D_{\langle a}D_{b\rangle} f$$

here (u, x^A) are affine coordinates on Scri, and $\xi = f\partial_u + Y^A\partial_A$ are symmetry vector fields

$$\dot{f} = T + \tfrac{u}{2}\mathscr{D}Y$$

Identifying charges

When applied to the BMS symmetries, we obtain the Ashtekar-Streubel flux

$$Q_\xi^{\rm BMS}[S_2] - Q_\xi^{\rm BMS}[S_1] \overset{?}{=} F_\xi^{\rm BMS} = -\frac{1}{16\pi G}\int_{S_1}^{S_2} N_{ab}\delta_\xi \sigma^{ab}\epsilon_{\mathscr{I}}$$

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Identifying charges

$$I_\xi \omega = \delta j_\xi - d i_\xi \theta \; = d(\delta q_\xi - i_\xi \theta)$$

freedom: $\bar{\theta} = \theta + \delta \ell$

$\bar{q}_\xi = q_\xi + i_\xi \ell$

Two alternative procedures:

- bootstrap the charges from knowledge of Komar and ℓ
- `integrate' the fluxes

The first seems easier, but it is actually more subtle than one may think,
because it requires choosing an extension of the symmetry vector fields in the bulk,
and carefully subtracting spurious terms that choices of extensions typically introduce

In the examples studied, the results match (BMS, NEH, general null hypersurface)

Identifying charges

Applied to Scri, the procedure gives:

$$Q_\xi^{\rm BMS}[S_2] - Q_\xi^{\rm BMS}[S_1] \hat{=} F_\xi^{\rm BMS} = -\frac{1}{16\pi G}\int_{S_1}^{S_2} N_{ab}\delta_\xi \sigma^{ab}\epsilon_{\mathscr{I}}$$

$$\delta_\xi \sigma_{ab} = (\pounds_\xi - \dot{f})\sigma_{ab} + D_{\langle a}D_{b\rangle} f$$

here (u, x^A) are affine coordinates on Scri, and $\xi = f\partial_u + Y^A\partial_A$ are symmetry vector fields

$$\dot{f} = T + \tfrac{u}{2}\mathscr{D}Y$$

$$Q_\xi^{\rm BMS} = \frac{1}{8\pi G}\oint_S (2fM_\rho + Y^A P_A)\epsilon_S$$

$$M_\rho := -{\rm Re}(\psi_2 - \sigma N) = M + \tfrac{1}{4}\rho_{ab}\sigma^{ab}$$

Geroch super-momentum '77

$$m^A P_A = -\left(\psi_1 + \sigma\eth\bar{\sigma} + \frac{1}{2}\eth(\sigma\bar{\sigma})\right)$$

Dray-Streubel angular momentum '84

Energy-loss recovered and generalized

$$Q_\xi^{\rm BMS}[S_2] - Q_\xi^{\rm BMS}[S_1] \hat{=} F_\xi^{\rm BMS} = -\frac{1}{16\pi G}\int_{S_1}^{S_2} N_{ab}\delta_\xi \sigma^{ab}\epsilon_{\mathscr{I}}$$

$$\xi = f\partial_u + Y^A\partial_A$$

$$f = T + \tfrac{u}{2}\mathscr{D}Y$$

Specializing to Y=0:

$$\oint_{S_2} TM\epsilon_S - \oint_{S_1} TM\epsilon_S = -\frac{1}{32\pi G}\int_{\mathcal{N}} T(N_{ab}N^{ab} - 4D_aD_bN^{ab})\epsilon_{\mathcal{N}}$$

`hard flux' `soft flux' (memory effects)

- For *T=1* we recover Bondi's energy loss formula,
- For arbitrary *T* it is generalized to arbitrary flux of super-momentum

Specializing to T=0:

- For DY=0 we have the angular momentum flux, and for pure DY the center-of-mass flux

A beautiful synergy of results

Summarizing, it is possible to obtain fluxes and charges at Scri using two different methods:

- the Wald-Zoupas procedure
- the Hamiltonian integration of Ashtekar-Streubel '81, refined to finite regions of Scri in our paper

in both cases identifying the fluxes is the `easy' part, and finding the charges requires more work

I didnt have time to cover: the geometrical description of Scri can be entirely given in terms of weakly isolated horizons (Ashtekar-Fairhurst-Krishnan,Lewandowski,Booth,Khera,Kolanowski…)

This leads to a common description of both Scri and black hole/cosmological horizons,
even though they are physical completely different!

$$\dot{\bar{D}}_a \bar{\jmath}_b \hat{=} \bar{D}_a \bar{\omega}_b + \bar{\omega}_a \bar{\omega}_b + \bar{k}^c \, \bar{C}_{c\underset{\leftarrow}{a}b}{}^d \, \bar{\jmath}_d + \frac{1}{2} \left(\bar{S}_{\underset{\leftarrow}{a}b} + (\alpha - \frac{1}{6} \bar{R}) \, \bar{q}_{ab} \right)$$

The key to the common language yet completely different physics is that while they share the same geometry, black hole and cosmological horizons satisfy Einstein's equations, whereas at Scri it is the *conformal* Einstein's equations that are valid

In a domain as thorny as that of observables in general relativity,
it is quite remarkable to find such a strong synergy of results, with different methods being applicable to very different systems, and giving the same answers!

III - Localised energy

Energy-loss and memory

Let us look again at the super-translation flux-balance law:

$$\oint_{S_2} TM\epsilon_S - \oint_{S_1} TM\epsilon_S = -\frac{1}{32\pi G}\int_{\mathcal{N}} T(N_{ab}N^{ab} - 4D_aD_bN^{ab})\epsilon_{\mathcal{N}}$$

T is an arbitrary function on the sphere; choosing a Bondi frame, we can decompose it in spherical harmonics; then l=0,1 correspond to the 4 global translations, and l>1 are super-translations (this characterisation is foliation-dependent)

The soft term vanishes for l=0,1: the global translations have a **strictly negative flux** and this is crucial for the unambiguous interpretation of $\oint M\epsilon_S$ as the total energy

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Consider then a profile for T peaked on some given point of the sphere:
it will contain arbitrary l>1 modes, hence the corresponding mass multipole does **NOT** have a negative-definite flux

⇨ for a gravitational system, it is only possible to characterise the ***global*** loss of energy (and momentum). **Localized energy density appears to be still an elusive concept,** even with the additional universal structure available at Scri

How about Bondi's bucket then? can't we identify how much energy flows through it and be available to heat it up?

Modified super-momentum charge

$$\oint_{S_2} TM\epsilon_S - \oint_{S_1} TM\epsilon_S = -\frac{1}{32\pi G}\int_{\mathcal{N}} T(N_{ab}N^{ab} - 4D_aD_bN^{ab})\epsilon_{\mathcal{N}}$$

At this point, observe that the soft term is a total time derivative:

it is actually possible to move it on the LHS and include it in the definition of the charge!

$$\oint T(M + D_aD_b\sigma^{ab})\epsilon_S$$

The resulting modified super-momentum would have negative energy flux for every mode

It would then be possible to talk about energy loss of a localised multipole in an unambiguous way

Such a modification has been considered in the past (Dain-Moreschi '02, `nice cuts')

but the obvious problem is that the resulting charges are not Hamiltonian.

We have, after all, just argued that they are identified *uniquely*…

Symplectic freedom

The fluxes and charges we constructed are indeed unique, once the symplectic structure is given

But the field equations only determine the symplectic 2-form up to corner terms

- examples:
  EH vs ADM (Brunett-Wald '90)
  metric vs tetrads (De Paoli-S '16)
- see general discussion in Harlow-Wu, Freidel-Geiller-Pranzetti, Margalef-Bentabol-Barbero-Villasenor, …
- Used also for extended BMS (`super-rotations') (Campiglia-Pereza '20)
  (see Campiglia's talk at Loops '22)

Is there a `corner deformation' of the symplectic 2-form that turns the standard BMS charges into these ones with strictly negative flux?

We claim that the answer is yes

To construct it, we need to disentangle the vacuum structure of the phase space

Vacuum directions

Ashtekar '81

The radiative phase space at *Scri* can be described using connections

but the connection knows more than just the radiation: also the shear associated with any cross section of Scri

(shear occurs also in vacuum: shearing null geodesic congurences in Minkowski)

at Scri, any super-translation creates shear without the need for radiation

→ **vacuum directions**

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→ **vacuum directions**

Picking a choice of origin, say zero shear for a chosen cross-section,
we can parametrize all vacuum connections as a certain operator acting on a choice of `bad cut'

$$u_0 = u_0(x^A)$$

aka super-translation field (Compere-Fiorucci-Ruzziconi) aka super-translation Goldstone (Strominger)

Using this parametrization, we can define what was called `covariant' shear in CFR'18 :

$$\mathcal{S}_{ab} := (D_{\langle a} - \mathring{D}_{\langle a})l_{b\rangle} = \sigma_{ab} - \mathring{\sigma}_{ab} = \sigma_{ab} - \frac{u}{2}\rho_{ab} + \left(D_{\langle a}D_{b\rangle} + \frac{1}{2}\rho_{\langle ab\rangle}\right)u_0$$

Localized energy density

$$\mathcal{S}_{ab} := (D_{\langle a} - \overset{\circ}{D}_{\langle a})l_{b\rangle} = \sigma_{ab} - \overset{\circ}{\sigma}_{ab} = \sigma_{ab} - \frac{u}{2}\rho_{ab} + \left(D_{\langle a}D_{b\rangle} + \frac{1}{2}\rho_{\langle ab\rangle}\right)u_0$$

Crucially, $\delta_\xi \mathcal{S}_{ab} \neq \delta_\xi \sigma_{ab}$ (because $\delta_\xi u_0 \neq 0$)

transforms homogeneously under super-translations: $\delta_T \mathcal{S}_{ab} = T N_{ab}$

then obviously $\theta \sim N_{ab}\delta_\xi \sigma^{ab} \neq N_{ab}\delta_\xi \mathcal{S}^{ab}$

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transforms homogeneously under super-translations: $\delta_T \mathcal{S}_{ab} = T N_{ab}$

then obviously $\theta \sim N_{ab}\delta_\xi \sigma^{ab} \neq N_{ab}\delta_\xi \mathcal{S}^{ab}$

However, there exists a (unique) corner modification ϑ such that $\theta + d\vartheta = N_{ab}\delta_\xi \mathcal{S}^{ab}$

The resulting flux contains only the hard term

The resulting charge is ***not*** the one proposed by Dain-Moreschi, which we prove is not covariant, but contains extra terms that depend explicitly on the vacuum solution

Conclusions

In the paper with Abhay, we showed that the geometry of Scri can be put on the same footing as that of black hole and cosmological horizons: they are all examples of weakly isolated horizons

- Their dramatically different physics can be understood from the different implementation of Einstein's equations
- Unified description for fluxes and charges
- The selection of fluxes and charges that comes from a preferred symplectic potential in the Wald-Zoupas approach can also be understood as a choice of topology on the radiative phase space

While the fluxes and charges are unique once the symplectic structure is given,
we can entertain the possibility of modifying it by a corner term
In the paper with Antoine, we showed that by doing so, it is possible to modify the super-momentum charge so that the flux is negative-definite for all multipoles

- There is a unique covariant corner term that can be added to the radiative phase space so that the flux contains only the hard term
- This provides an avenue for a notion of localized energy density of GWs