Progress in twisted geometries and spin foam transition amplitudes

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Centre de Physique Théorique

Not easy to give a talk and stay focused on work on this particular day and what it could bring down on us…

To help you stay distracted, *I have prepared a very **dense** and **long** seminar*

But fear not too much, some technical slides will be skipped and are here only for your reference if you want to later on look at some details

Goal of the research presented

- Improve our control of the semiclassical geometry of LQG on a fixed graph
- Improve calculations with Lorentzian spin foam amplitudes

Outline

I will give a brief review of twisted geometries, and cover three recent results:

1. Lorentzian extrinsic geometry from SU(2) data

2. An analysis of conformal transformations on twisted geometries

3. A scheme to efficiently compute transition amplitudes with the EPRL Lorentzian model

… Let’s twist again!

LQG phase space on a fixed graph

GR phase space

$$\begin{tikzcd} \{q_{ab}, K^{ab}\} \arrow[r] \arrow[d] & \{A^i_a, E^a_i\} \arrow[d] \\ (\ell_e, \theta_t) \arrow[r, hook] & \{X_l, g_l\} \Leftrightarrow \{j_l, \bar{\xi}_l, Z_n\} \end{tikzcd}$$

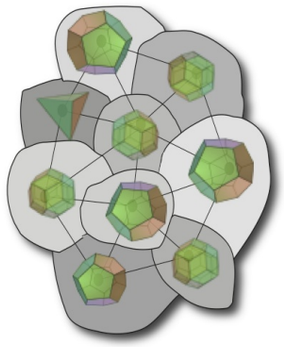
continuous geometry

discrete geometry

$$A^i = \gamma\omega^{0i} - \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}(E) \;\Rightarrow K^a_i = K^a_i(E, A)$$

Truncation:

Lack of perturbative expansion on a given background: for practical computational purposes we typically resort to a *truncation* of the theory to a finite number of dofs: a fixed, finite graph
(cfr. a QFT with fixed number of external legs)



Twisted geometry: a collection of polyhedra described by areas and angles,
plus extrinsic geometry encoded in the SU(2) holonomies

Truncated phase space on a fixed graph
$T^*SU(2)^L//C_n$

What is the discrete extrinsic geometry in terms of the SU(2) data?

$$\theta_l = \theta_l(X_i, g_i)$$

$$\Updownarrow$$

$$K^a_i = K^a_i(E, A)$$

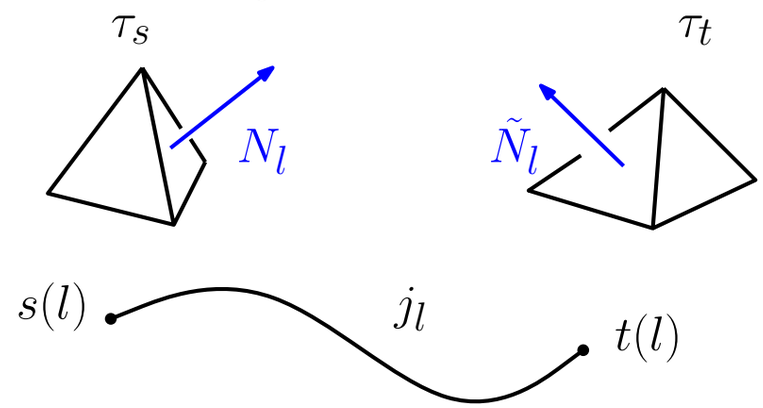
(It will play an important role in understanding dilatations)

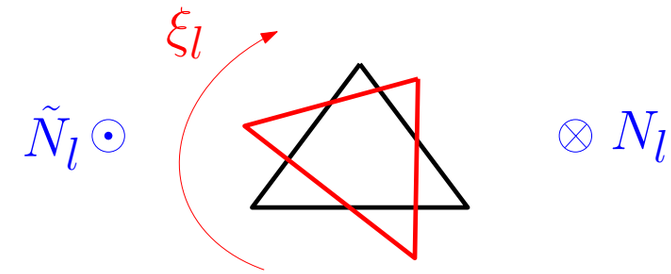
Twisted geometries

<u>On a single link:</u>

$$T^*SU(2): \ (X,g) \ \Rightarrow \ (X,\tilde{X},\xi) \qquad j := X^2 = \tilde{X}^2 \ \Rightarrow \ (N,\tilde{N},j,\xi)$$

$$\tilde{X} = -gXg^{-1}$$





(loosely switching here between X as vectors and as matrices using $\sigma_i^A{}_B$ isomorphism)

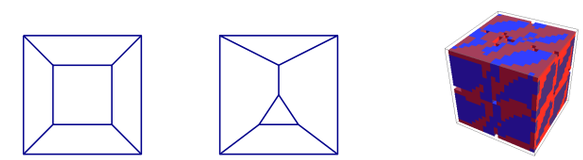
<u>On the graph, gauge-invariant phase space:</u> $T^*SU(2)^L//C_n$

$$6L - 6N = 2L + \underbrace{4L - 6N}_{Z_n} \qquad \sum_{\text{nodes}} 2(\text{val}_n - 3)$$

$(j_l, \bar{\xi}_l)$

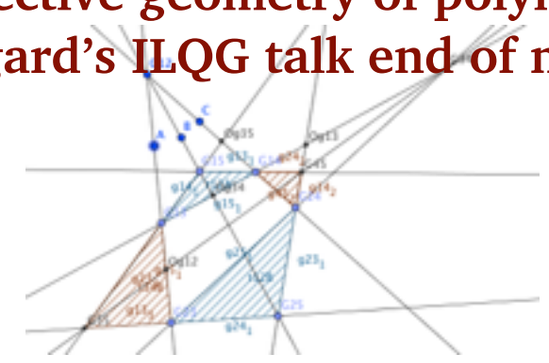
(cross ratios, or KM, or normals, or angles, or triple products)

(Livine-Freidel-Krasnov '08)



(Bianchi-Donà and S '10) 3d slice of $\mathcal{S}_6$, cuboids blue

The projective geometry of polyhedra, see Haggard's ILQG talk end of november



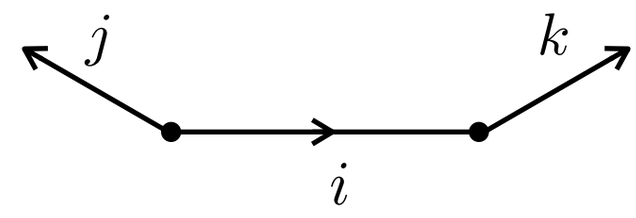
$\bar{\xi}_l$ *(non-local)* gauge-invariant quantity built from g_l

Two different strategies: (both cases provide redundant parametrizations)

- use Wilson loops (Livine et al.)
- use geometric framework (Freidel-Hnybida '13)

$$\xi_f = \frac{1}{2}\arccos\left(\text{Tr}\, g_f\right)$$

$$\cos\xi^i_{jk} := \frac{(\tilde{X}_i \times X_k)\cdot g_i \triangleright (X_i \times X_j)}{|\tilde{X}_i \times X_k|\,|X_i \times X_j|}$$



Extrinsic geometry in terms of SU(2) data

Hence, SU(2) gauge-invariant phase space describes flat polyhedra with a twist angle among them $\{j_i, \xi^i_{jk}, Z_n\}$

What is the discrete extrinsic geometry in terms of the SU(2) data?

$$A^i = \gamma\omega^{0i} - \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}(E) \;\Rightarrow K^a_i = K^a_i(E, A) \quad \longrightarrow \quad \boxed{\theta_l = \theta_l(X_i, g_i)}$$

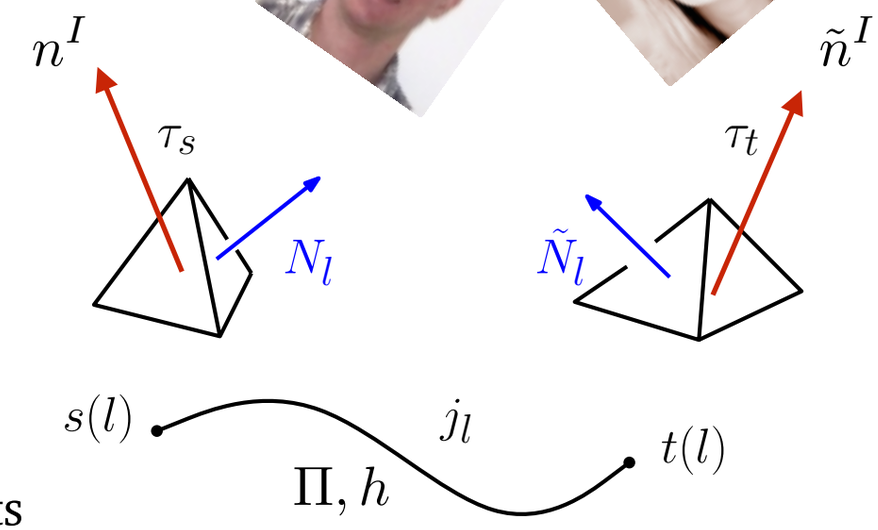
Precisely like in the continuum, it requires an embedding in the Lorentzian phase space, provided by the secondary simplicity constraints

1. Identify discrete Lorentzian extrinsic curvature; $\Longrightarrow$ definition of the boost dihedral angle θ
2. Relate it to SU(2) data; $\Longrightarrow$ definition of discrete secondary constraints:

   the (γ-depending) embedding
   of AB SU(2) in Lorentzian phase space

Two different angles on the same quantity

Consider the covariant phase space, on each link: $(\Pi, h) \in T^*SL(2,\mathbb{C})$



Two different definitions for the boost dihedral angle:

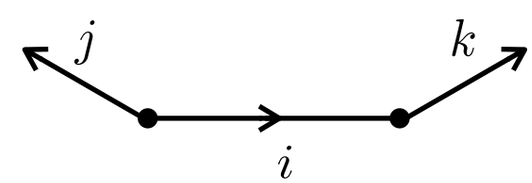
1. gauge-fixed: **Boost among 4d normals to the polyhedra**
(Wieland and S '12)

$$\cosh \Xi_i := -\tilde{n}_I \Lambda^I{}_J(h_i)\, n^J$$

$\Xi = \Xi(X, g)$ from stabilising primary simplicity constraints

2. gauge-invariant: **Boost among self-dual edge vectors**
(Dittrich-Ryan '08, here generalised to Lorentzian signature and arbitrary cellular decomposition)

$$\cosh \theta^i_{jk} := \frac{(\tilde{\Pi}_i \times \Pi_k) \cdot h_i \triangleright (\Pi_i \times \Pi_j)}{\sqrt{(\tilde{\Pi}_i \times \Pi_k)^2}\,\sqrt{(\Pi_i \times \Pi_j)^2}}$$



$\theta = \theta(X, g)$ from discretising Levi-Civita condition

The price to pay for gauge-invariance is linear dependence:

- As many angles per link as its valence; pick one representative to parametric the phase space
- Only when shapes match, they all coincide

On-shell of the (primary) simplicity constraints: $\theta^i_{jk} = \Xi_i + i(\gamma \Xi_i - \xi^i_{jk})$

The covariant picture: embedding T*SU(2) in T*SL(2,C)

SKIP

Two spinors describe SU(2) phase space: (cfr. Schwinger rep.) (Laurent and S '10)

$$\mathbb{C}^4//C \simeq T^*SU(2) \qquad (z^A, \tilde{z}^A)\,//\,C = ||z||^2 - ||\tilde{z}||^2 = 0 \quad \longrightarrow \quad \{X, g\} \Leftrightarrow \{j, \xi, N, \tilde{N}\}$$

(At the quantum level, $\mathcal{H} = \oplus(V_j \otimes \bar{V}_j)$: without it, holonomies don't commute and algebra does not close)

Two twistors describe SL(2,C) phase space: (cfr. Penrose rep.) (Wolfgang and S '12)

$$\mathbb{C}^8//\mathcal{C} \simeq T^*SL(2,\mathbb{C}) \qquad (Z^\alpha, \tilde{Z}^\alpha)\,//\,\mathcal{C} = [\pi|\omega\rangle - [\tilde{\pi}|\tilde{\omega}\rangle \quad \longrightarrow \quad \{\Pi, h\} \qquad Z^\alpha = \begin{pmatrix} \omega^A \\ i\bar{\pi}_{\dot{A}} \end{pmatrix}$$

(matches dilatations and helicities: $\pi\omega = D + is \quad D = \tilde{D}, \quad s = \tilde{s}$)

Simplicity constraints *S*:

$$K + \gamma L = 0 \qquad \omega^A = \frac{||\omega||^2}{(\gamma - i)j}\delta^{A\dot{A}}\bar{\pi}_{\dot{A}}$$

$$T^*SL(2,\mathbb{C})//S = T^*SU(2)$$

Only first class part:
$D - \gamma s = 0$
Orbits spanned by Ξ

We then compute:

$$\cosh \Xi_i := -\tilde{n}_I \Lambda^I{}_J(h_i)\, n^J \approx \ln \frac{||\omega_i||^2}{||\tilde{\omega}_i||^2}$$

$$\cosh \theta^i_{jk} := \frac{(\tilde{\Pi}_i \times \Pi_k) \cdot h_i \triangleright (\Pi_i \times \Pi_j)}{\sqrt{(\tilde{\Pi}_i \times \Pi_k)^2}\sqrt{(\Pi_i \times \Pi_j)^2}} = \frac{1}{2}\frac{[\omega_i|\pi_j\rangle\,[\omega_i|\omega_j\rangle\,[\tilde{\pi}_i|\omega_k\rangle\,[\tilde{\pi}_i|\pi_k\rangle + [\pi_i|\omega_j\rangle\,[\pi_i|\pi_j\rangle\,[\tilde{\omega}_i|\pi_k\rangle\,[\tilde{\omega}_i|\omega_k\rangle}{\sqrt{[\omega_i|\omega_j\rangle[\pi_i|\pi_j\rangle[\omega_i|\pi_j\rangle[\pi_i|\omega_j\rangle}\sqrt{[\omega_k|\tilde{\omega}_i\rangle[\pi_k|\tilde{\pi}_i\rangle[\omega_k|\tilde{\pi}_i\rangle[\pi_k|\tilde{\omega}_i\rangle}$$

$$\approx \cosh\left(\Xi_i + i(\gamma\Xi_i - \xi^i_{jk})\right)$$

(Langvik and S '16)

Secondary constraints

In the time gauge, the boost Ξ between the normals is related to Dittrich-Ryan θ by the SU(2) twist angle among the edge vectors :

$$\theta^i_{jk} = \Xi_i + i(\gamma\Xi_i - \xi^i_{jk})$$

Furthermore, the two definitions of secondary constraints on shape matched configurations coincide:

1. From stabilising flat Hamiltonian constraint (Anzà and S '14) $\gamma\Xi_i - \xi^i_{jk} = 0$

2. From discretising Levi-Civita condition à la Regge (Dittrich-Ryan '08) $\theta^i_{jk} = \bar{\theta}^i_{jk}$

Then: $\Xi_i = \frac{1}{\gamma}\xi^i_{jk}(X_l, g_l)$ provides the (γ-depending) embedding of AB SU(2) in Lorentzian phase space $\theta_l = \theta_l(X_i, g_i)$

By construction, both techniques imply the shape matching conditions

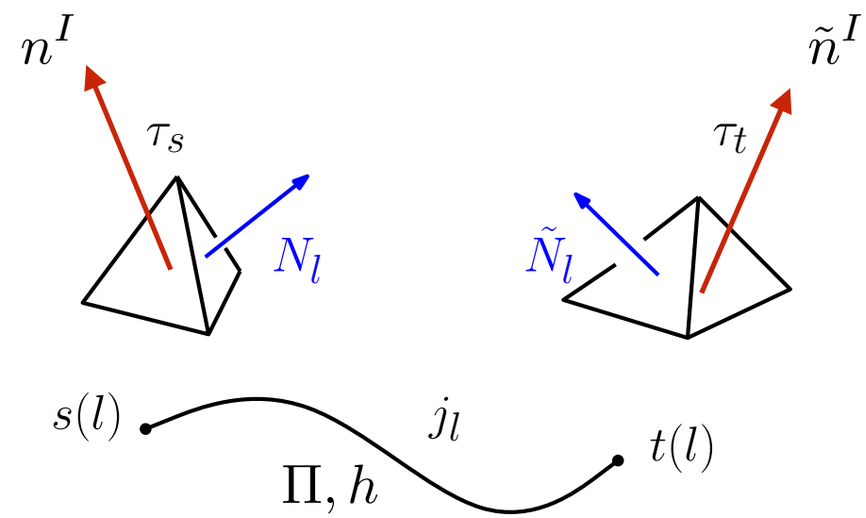
An edge-dependent off-shape-matching Levi-Civita condition was defined in Haggard-Rovelli-Vidotto-Wieland '13

can be used to define boost angle and AB embedding off the shape matching conditions
(to appear, with Bianca and Wolfgang)

(alternative: use Freidel-Ziprick's spinning geometry picture)

Summary

SU(2) phase space	SL(2,C) phase space
(g_l, X_l)	(h_l, Π_l)
ξ_i twist angle	Ξ_l boost in time gauge
ξ^i_{jk} edge vector's angles	θ^i_{jk} edge bi-vector's angles
	$\theta^i_{jk} = \Xi_i + i(\gamma\Xi_i - \xi^i_{jk})$



Extrinsic geometry in terms of reduced data:

- for Regge configurations $\Xi_i = \frac{1}{\gamma}\xi^i_{jk}(X_l, g_l)$
- for non-shape-matching, to appear

Note: in practise, it is often convenient to fix also SU(2) gauge and align edges, so to have $\xi^i_{jk} \equiv \xi_i$

Then, Regge's $\Xi_i = \frac{1}{\gamma}\xi_i$

(used for instance in computing the BH tunnelling time)

At the level of the holonomy, we have : $g = \cosh[(i\gamma+1)\frac{\Xi}{2}]\, h + \sinh[(i\gamma+1)\frac{\Xi}{2}]\, \hat{h}$

discrete equivalent of AB connection: $iA^i = i(\gamma\omega^{0i} - \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}) = (i\gamma+1)\omega^{0i} + \omega^i_{SD}$

Remark: Twistor's incidence relation

SKIP

In twistor terms, the primary simplicity constraints are a restriction on the incidence relation

$$\omega^A = iX^{A\dot{A}}\bar{\pi}_{\dot{A}} \qquad X^{A\dot{A}} = n^{A\dot{A}}\,\frac{j\sqrt{1+\gamma^2}}{||\pi||^2}\,e^{i\arctan 1/\gamma}$$

- Phase shift from γ making twistors not null: $D - \gamma s = 0$
- Direction n from linear simplicity constraints

naturally extends to <u>space-like</u> and <u>null</u> reductions (Rennert, Zhang)

- Reduction to rotation subgroup SU(2):

$$K = 0$$

Twistor's spinors related by a parity

$$\omega^A = i\frac{||\omega||^2}{j}\delta^{A\dot{A}}\bar{\pi}_{\dot{A}}$$

- Reduction to AB auxiliary SU(2):

$$K + \gamma L = 0$$

Twistor's spinors related by a parity and a phase

$$\omega^A = \frac{||\omega||^2}{(\gamma - i)j}\delta^{A\dot{A}}\bar{\pi}_{\dot{A}}$$

Simple idea: use twistors to make causality *dynamical*, as often advocated by Immirzi
…for another talk…

II. Conformal transformations

M. Langvik and S, Twisted geometries,
twistors and conformal transformations, arXiv:1602.01861

M. Dunajski, M. Langvik and S, A self-dual octahedron
and conformal symmetry breaking, to appear

The interest in SU(2,2) conformal transformations

Some physical questions we are interested in:

- is there an analogue of the conformal splitting (scale+causal structure) for quantum geometry?
- is there an analogue of the conformal instability in spin foams?
- is there a way to understand LQG as the softly broken phase of a conformal invariant theory?

My discussion will be limited to results for the classical theory on a fixed graph

There, curved geometry is described à la Regge from piecewise-flat chunks
$\Rightarrow$ *the local action of the Minkowski conformal symmetry group SU(2,2) may be relevant to these questions*

Some obvious difficulties in studying conformal transformations on discrete geometries:

- Regge calculus: when edge lengths are stretched, angles change $\delta\ell_{xy} = (\lambda_x + \lambda_y)\ell_{xy}$
  $\Rightarrow$ no conformal meaning possible beyond first linearised order

- Holonomy-fluxes: a phase space with compact directions (g, X)
  $\Rightarrow$ no natural definition of dilatations

An interesting way forward: the conformal transformations induced from twistor space

Discretising the continuum dilatation generators

Before getting to twistors, how far can we get by simply discretising the continuum phase space dilatation generator, for instance? *(Note: this is also the generator of canonical transformations introducing γ)*

$$W = E^a_i A^i_a, \qquad \exp(\lambda W) \triangleright A^i_a = e^{-\lambda} A^i_a, \qquad \exp(\lambda W) \triangleright E^i_a = e^{\lambda} E^i_a.$$

Discretisation: $W = 2\mathcal{L}_X \chi^{(\frac{1}{2})}(g) = -2\mathrm{Tr}(gX)$

Infinitesimal action on holonomies and fluxes: $\{W, g\} = \mathbb{1} - \frac{\mathrm{Tr} g}{2} g, \qquad \{W, X^i\} = \frac{1}{2}\mathrm{Tr} g\, X^i - \epsilon^{ijk} X^j \mathrm{Tr}(\tau^k g).$

To evaluate finite action, much faster to use spinors: $e^{\lambda W} \triangleright \begin{pmatrix} z^A \\ \tilde{z}^A \end{pmatrix} = \begin{pmatrix} \cosh\lambda/2 & \sinh\lambda/2 \\ \sinh\lambda/2 & \cosh\lambda/2 \end{pmatrix} \begin{pmatrix} z^A \\ \tilde{z}^A \end{pmatrix}$

Then:

$$g_W := e^{\lambda W} \triangleright g = \frac{1}{\cosh\lambda + \frac{\mathrm{Tr} g}{2}\sinh\lambda}\left[\cosh^2\frac{\lambda}{2} g + \sinh^2\frac{\lambda}{2} g^{-1} + \sinh\lambda \mathbb{1}\right],$$

$$X^i_W := e^{\lambda W} \triangleright X^i = X^i + \left(\sinh^2\frac{\lambda}{2} + \frac{X^2 \mathrm{Tr} g}{2(X^2 - X\cdot\tilde{X})}\sinh\lambda\right)(X^i - \tilde{X}^i) - \frac{\mathrm{Tr}(gX)}{X^2 - X\cdot\tilde{X}}\sinh\lambda\, \epsilon^{ijk} X^j \tilde{X}^k.$$

Properties:

- preserves the Poisson structure
- reproduces (by construction!) continuum limit: $g_W \simeq \mathbb{1} + e^{-\lambda} A, \qquad X^i_W \simeq e^{\lambda} X_i$
  preservation of angles, rescaling of areas, volumes
- **breaks closure!** $e^{\lambda W} \triangleright \sum_{l\in n} \vec{X}_l = \sum_{l\in n} \vec{X}_{Wl} \neq 0, \quad \Rightarrow$ *no geometric meaning on the gauge-invariant phase space*

Dilatations from twistor space

An alternative definition of dilatations can be obtained from twistors

We already used the fact that twistor carry a representation of the Lorentz algebra to parametric Lorentzian holonomies and fluxes

$$\mathbb{T}^2//C \simeq T^*SL(2,\mathbb{C})$$

Twistor space carries also a representation of the Minkowski conformal group SU(2,2):

$$J^{IJ} = \omega^{(A}\pi^{B)}\epsilon^{\dot{A}\dot{B}} + \text{cc}, \qquad P^I = i\pi^A\bar{\pi}^{\dot{A}}, \qquad C^I = i\omega^A\bar{\omega}^{\dot{A}}, \qquad D = \text{Re}(\pi_A\omega^A).$$

Dilatation algebra: $\{D, J^{IJ}\} = 0, \qquad \{D, P^I\} = -P^I, \qquad \{D, C^I\} = C^I$

Since twisted geometries can be described by twistors, it is natural to ask what transformations are induced by this SU(2,2) conformal action

- ***P* and *C* do not commute with the geometric constraints, $\Rightarrow$ no direct geometric interpretation**

  Full conformal invariance is not broken by the infinity twistor (which reduces to Poincaré subgroup) *but by the area-matching condition* $D = \tilde{D}$ (and which reduces to Lorentz+dilatations)

- ***D* commutes with everything, its action is well defined on *T*SU(2)***

What is its geometric meaning?

$$\{D, \Xi\} = 1, \qquad \{D, \xi\} = \gamma, \qquad \{s, \Xi\} = 0, \qquad \{s, \xi\} = 1.$$

The answer is quite amusing:
- *preserves all scales*
- shifts the dihedral angle

$$e^{\mathcal{D}} \triangleright \Xi = \Xi + u, \qquad e^{\mathcal{D}} \triangleright \xi = \xi + \gamma u.$$

$$e^{\mathcal{D}} \triangleright \cosh\theta^i_{jk} = \cosh(\theta^i_{jk} + u_i).$$

Twistor's dilatations on twisted geometries

- *P* and *C* incompatibles with geometric constraints
- *D* is compatible: its action preserves the scales and shifts the extrinsic geometry (corresponds to a 1-parameter family of embeddings of *T*SL(2,C)* in twistor space)

Why are the dilatations generated by D so different from geometric dilatations?
because these are dilatations in the twistor phase space, not in the holonomy-fluxes phase space

$$\exp(\lambda D) \triangleright |\omega\rangle = e^{\lambda/2}|\omega\rangle, \qquad \exp(\lambda D) \triangleright |\pi\rangle = e^{-\lambda/2}|\pi\rangle, \qquad \lambda \in \mathbb{R}$$

These are the dilatations wrt which the fluxes are just angular momentum generators in Minkowski

Interesting dynamical role: *changing the extrinsic curvature and not the intrinsic one means changing the bulk curvature (think of Gauss-Codazzi equation)*

Conclusions:

- Like in Regge calculus, discrete dilatations are not well defined for holonomy-fluxes/twisted geometries beyond first linearised order
- We can define a discretised dilatation operator *W*, whose finite action preserves the Poisson structure, but not closure
- We can consider the twistor dilatation *D*, compatible with all constraints, that instead of shifting scales, shifts the extrinsic curvature

<u>Remark:</u> *It would be interesting to compare these actions with squeezed states recently considered by Eugenio and collaborators (see also Livine, Marciano-Zhang) obtained by first dilating on the auxiliary spinorial space then projecting down to gauge-invariant holonomy-fluxes*

More thoughts on SU(2,2) breaking

Maybe skip?

- We have understood that the area matching is responsible for breaking the *su(2,2)* symmetry of twistor space to the *sl(2,C)* x *D* one of holonomies and fluxes

- **Can we build a full reduction** $T^*SU(2,2) \mapsto T^*SL(2,\mathbb{C})$ **?**

Embedding of Lorentzian spin nets into conformal spin nets
(one of the very first reasons Penrose introduced twistors!)

Need first to be able to describe *T*SU(2,2)* in twistor space…

The logic: generalise the dyadic representation of group elements used so far:

$(z^A, \delta^{A\dot{A}}\bar{z}_{\dot{A}}) \xrightarrow{g^A{}_B} (\tilde{z}^A, \delta^{A\dot{A}}\bar{\tilde{z}}_{\dot{A}})$	$(\omega^A, \pi_A) \xrightarrow{h^A{}_B} (\tilde{\omega}^A, \tilde{\pi}_A)$	$(Z_i^\alpha) \xrightarrow{G^\alpha{}_\beta} (\tilde{Z}_i^\alpha),\ i = 1..4$
Area matching	Complex area matching	helicity matchings+incidences
1 first class	2 first class	4 first class + 24 second class
$\mathbb{C}^4//C \simeq T^*SU(2)$	$\mathbb{C}^8//\mathcal{C} \simeq T^*SL(2,\mathbb{C})$	$(Z_i, \tilde{Z}_i) \in \mathbb{C}^{32}//C \simeq T^*U(2,2)$
(Laurent and S '10)	(Wolfgang and S '12)	(Dunajski, Langvik and S, to appear)

Unitarity as a self-dual octahedron

Maybe skip?

Logic: take two basis sets $(Z_i^\alpha, \tilde{Z}_i^\alpha)$

- Build the *su(2,2)* generators by linearity on each basis
- Build an *SU(2,2)* matrix mapping one basis to the other

The required new constraints are the orthogonality of the basis vectors Z_i^α with respect to $\eta = (++--)$

- Matching helicities $s_i = \tilde{s}_i$

(generate a U(4) algebra mixing the twistors)

- Incidences $\bar{Z}_i \cdot Z_j = 2 s_i \delta_{ij}$ (idem for tilded)

Constraint counting:
4 first class + 6x2x2 second class
= 32 dimensions removed

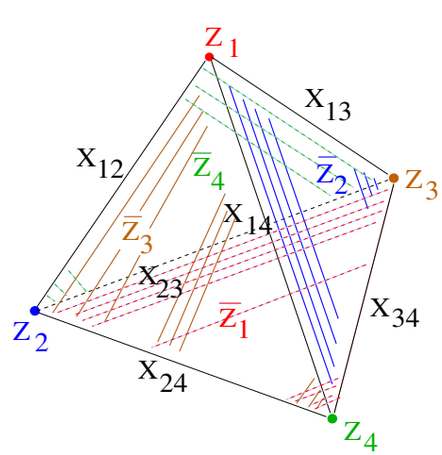
$$(Z_i, \tilde{Z}_i) \in \mathbb{C}^{32} // C \simeq T^* U(2,2)$$

(then impose det=1)

Meaning of the new constraints:
incidence of twistors

For non-null twistors: $\bar{Z} \cdot W = 0 \Rightarrow \begin{cases} W \text{ belongs to the plane } \bar{Z} \\ \alpha_W \cap \beta_Z \text{ on a complex null ray} \end{cases}$

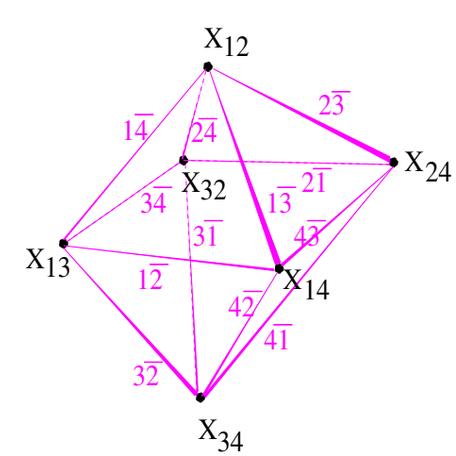
A tetrahedron in $P\mathbb{T}$



Self-dual in the sense that
a-planes intersections are
self-dual with β-plane intersections:

$$X_{ij} = \frac{1}{2} \epsilon_{ijkl} \bar{X}_{kl}$$

An octahedron in $M^{\mathbb{C}}$



Summary of classical parts I and II

1. Complete covariant discrete geometric picture:

- different definitions of extrinsic geometry and secondary constraints are related and comparable
- meaning of $g \in SU(2)$ as Ashtkar-Barbero clear;
- Lorentzian extrinsic curvature described by a boost related to SU(2) data from secondary constraints
- Interesting open questions remain in the non-shape-matched sector (torsion, action principle, etc)

2. Conformal transformations:

- Like in Regge calculus, discrete dilatations are not well defined for holonomy-fluxes/twisted geometries beyond first linearised order
- We can define a discretised dilatation operator *W*, whose finite action preserves the Poisson structure, but not closure
- We can consider the twistor dilatation *D*, compatible with all constraints, that instead of shifting scales, shifts the extrinsic curvature

3. T*SU(2,2) phase space

- Relevance for addressing important questions in the context of LQG
- Description in twistor space as symplectic sub-manifold corresponding to a self-dual octahedron
- Construction of SU(2,2) spin networks in progress, with the goal of studying whether interesting mechanisms of dynamical breaking of the symmetry can arise

Main lesson: Twistors very useful tools to perform explicit calculations, explore the geometry of loop quantum gravity on a fixed graph, and address existing/new questions

III. Dynamics: factorising the EPRL model

S, Boosting Wigner's nj-symbols, arXiv:1609.01632

Explicit evaluations of EPRL spin foam amplitudes

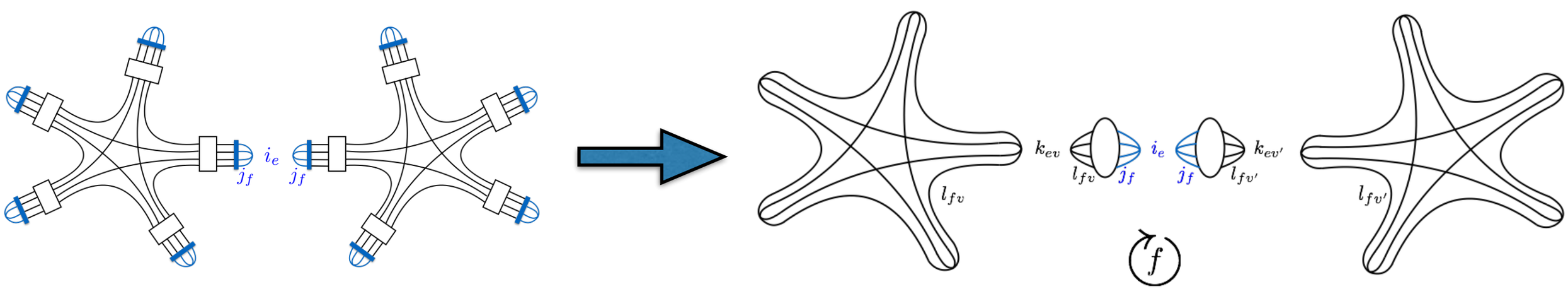
Cosmology, black hole physics, modified dispersion relations are making quantum gravity closer to observable physics, increasing the motivations to switch attention from formal model developments to performing actual calculations

Using the spin foam formalism for LQG dynamics, individual amplitudes extremely difficult to evaluate both analytically and numerically

Particularly true for Lorentzian signature, because of non compact groups,
associated unbounded group integrals, and less developed libraries of CG coefficients

The results I present here concern the explicit evaluation of the amplitudes for Lorentzian signature:
reduce the problem to a similar level of complexity as for Euclidean signature,
offer new tools for exact evaluations and semiclassical approximations

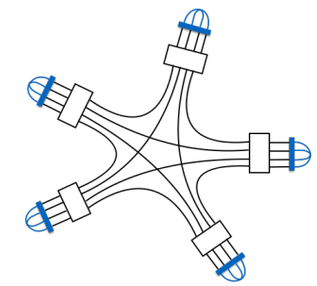
The evaluation scheme I will consider is the natural factorisation of SL(2,C) invariants in terms of SU(2) ones:



SL(2,C) group integrals SU(2) {15j} Boost integrals SU(2) {15j}

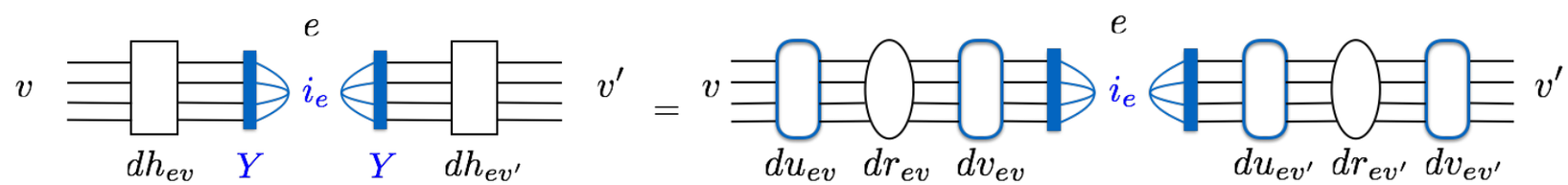
Factorisation of the EPRL model

$$Z_{\mathcal{C}} = \sum_{j_f, i_e} \prod_f (2j_f + 1) \prod_v A_v(j_f, i_e) \qquad A_5(j_f, i_e) =$$

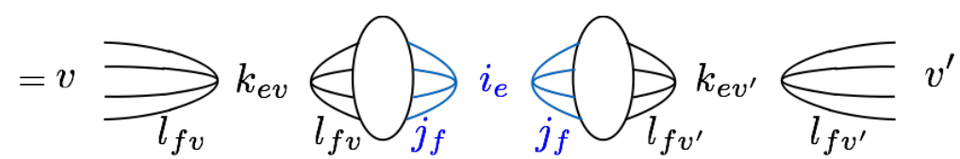


Use Cartan decomposition to split SL(2,C) integrals into integrals over SU(2) and a single boost *r*

$$h = u e^{\frac{r}{2}\sigma_3} v^{-1} \quad \Rightarrow$$

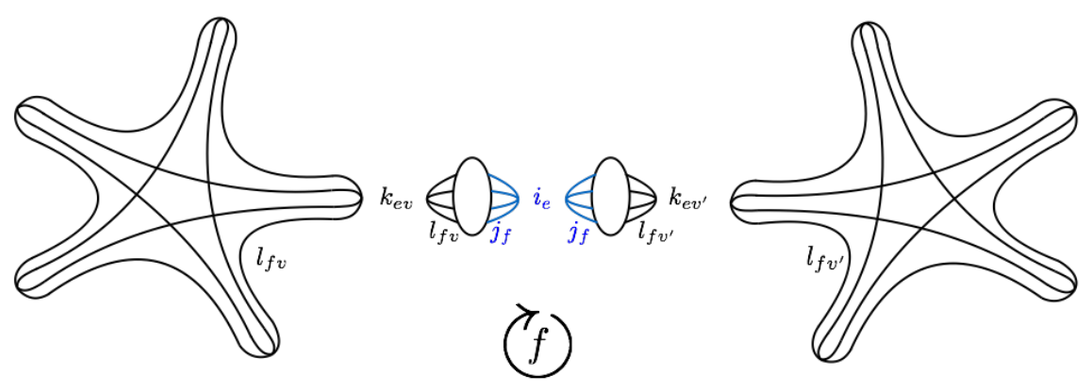


$$D^{(\rho,k)}_{jmln}(h) = D^{(j)}_{mp}(u) d^{(\rho,k)}_{jlp}(r) D^{(l)}_{pn}(v^{-1})$$



(11)

Performing the SU(2) integrals gives:

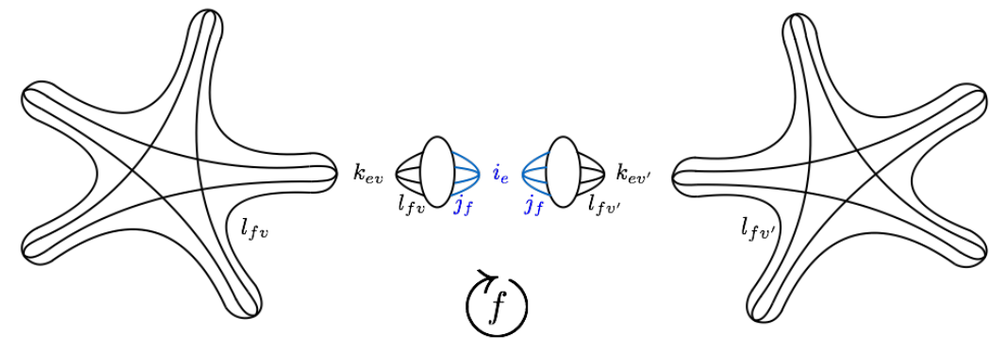


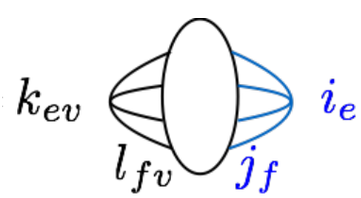
$$Z_{\mathcal{C}} = \sum_{j_f, i_e, l_{fv}, k_{ev}} \prod_f (2j_f + 1) \prod_e A^{\gamma}_e(j_f, i_e, l_{fv}, k_{ev}) \prod_v \{nj\}_v(l_{fv}, k_{ev})$$

With only boost integrals remaining in the dipole-like edge amplitudes

The edge and half-edge weights (aka dipole amplitudes)

$$Z_{\mathcal{C}} = \sum_{j_f, i_e, l_{fv}, k_{ev}} \prod_f (2j_f+1) \prod_e A_e^\gamma(j_f, i_e, l_{fv}, k_{ev}) \prod_v \{nj\}_v(l_{fv}, k_{ev})$$





$$B^\gamma(j_i, l_i; i, k) = \sum_{p_i} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix}^{(i)} \begin{pmatrix} l_1 & l_2 & l_3 & l_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix}^{(k)} \int d\mu(r) \left(\bigotimes_{i=1}^{4} d^{(\gamma j_i, j_i)}_{j_i l_i p_i}(r) \right)$$

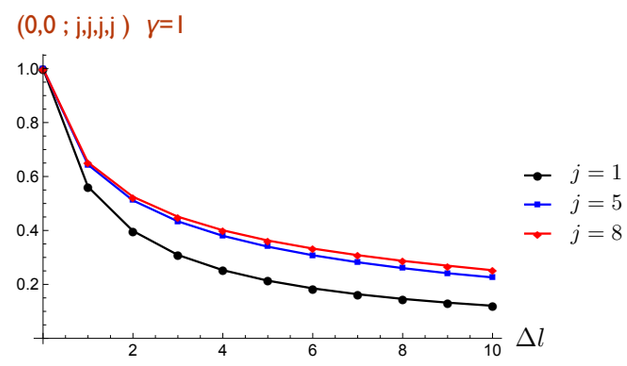
Geometric meaning: sum over boosts connecting two quantum tetrahedra
should have an intriguing semiclassical limit (wip with Pierre Martin-Dussaud)

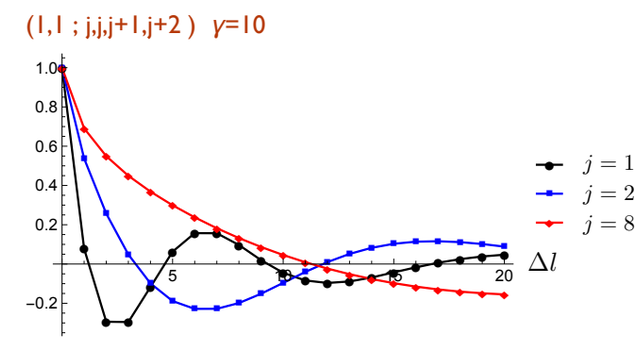
Some properties:

- **peakedness on minimal spins *l=j***

(numeric, no analytic formula yet)

$$\frac{B_4^\gamma(i,i;j;j+\Delta l)}{B_4^\gamma(i,i;j;j)} \sim \Delta l^{-1}$$





suggests definition of a ***simplified model*** with $l_{f_v} \equiv j_f$
much faster to evaluate, can be very useful for preliminary investigations, and relevant in some regimes

- simple asymptotic behaviour at minimal spins (Puchta '13)

(analytic via saddle point, numerically confirmed)

$$B_n^\gamma(i,k;Nj_i;Nj_i) \sim \frac{\delta_{ik}}{N^{3/2}} \frac{1}{[(1+\gamma^2)J]^{3/2}}$$

Note: saddle point at *r=0:*
no classical boost deforming shapes
without deforming areas

$\Rightarrow\ l > j$ <u>play important dynamical role</u>

- shows the role of γ in controlling peakedness
- virtual ρ off-shell of simplicity constraints

Relation to SL(2,C) Clebsch-Gordan coefficients

Unitary irreps of the principal series: $\rho \in \mathbb{R},\ k \in \mathbb{N}/2$

In Naimark's canonical basis, (J^2, J_z), CG coefficients factorise in terms of the SU(2) ones:

$$C^{\rho k j m}_{\rho_1 k_1 j_1 m_1 \rho_2 k_2 j_2 m_2} = \chi(\rho_1, \rho_2, \rho, k_1, k_2, k; j_1, j_2, j)\, C^{jm}_{j_1 m_1 j_2 m_2}$$

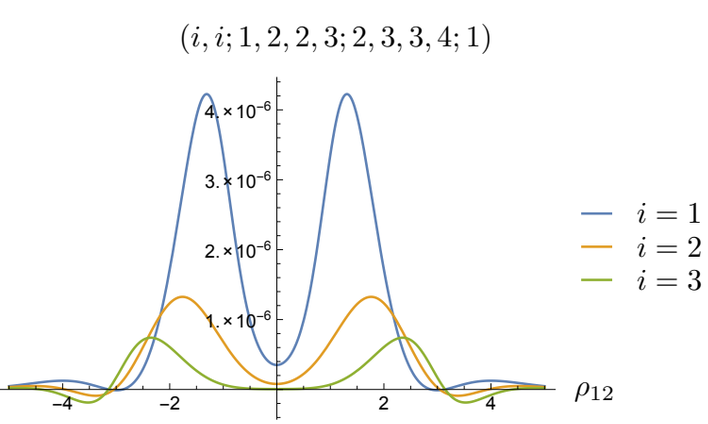
Half-edge weights are directly expressed in terms of the χ coefficients:

- 3-valent case (no intertwiners) :

$$B_3(\rho_i, k_i; j_i, l_i) = \int d\mu(r) \sum_{p_i} \begin{pmatrix} j_i \\ p_i \end{pmatrix} \begin{pmatrix} l_i \\ p_i \end{pmatrix} \bigotimes_{i=1}^{3} d^{(\rho_i, k_i)}_{j_i l_i p_i}(r) \quad \propto \quad \chi(\rho_i, k_i; j_i)\, \chi(\rho_i, k_i; l_i)$$

- 4-valent case (one intertwiner, quantum tetrahedron) :

$$B_4(\gamma j_i, j_i; j_i; l_i; j_{12}, l_{12}) = \int d\mu(r) \sum_{p_i} \begin{pmatrix} j_i \\ p_i \end{pmatrix}^{(j_{12})} \begin{pmatrix} l_i \\ p_i \end{pmatrix}^{(l_{12})} \bigotimes_{i=1}^{4} d^{(\gamma_i, j_i)}_{j_i l_i p_i}(r)$$

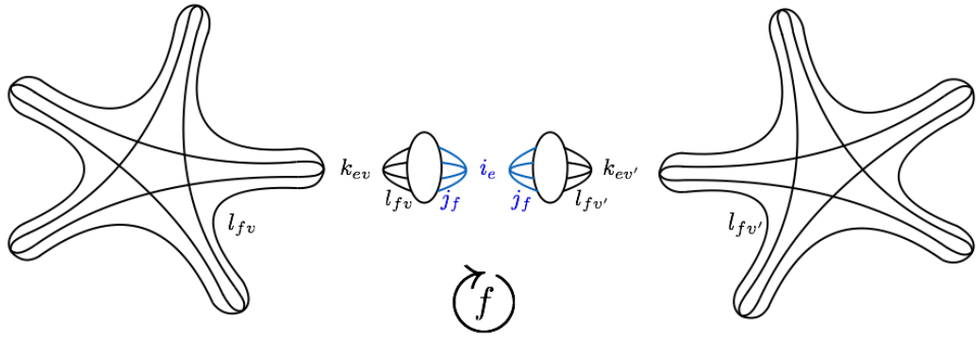


$$\propto \int_{-\infty}^{\infty} d\rho_{12} \sum_{k_{12}=-\min\{j_{12}, l_{12}\}}^{\min\{j_{12}, l_{12}\}} 4(\rho_{12}^2 + k_{12}^2)\, \bar{\chi}(j_1, j_2, j_{12}) \bar{\chi}(j_{12}, j_3, j_4) \chi(l_1, l_2, l_{12}) \chi(l_{12}, l_3, l_4).$$

Virtual irreps off-shell of the simplicity constraints but amplitudes peaked on on-shell values

Factorisation of the EPRL model: usefulness

$$Z_{\mathcal{C}} = \sum_{j_f, i_e, l_{fv}, k_{ev}} \prod_f (2j_f+1) \prod_e A_e^\gamma(j_f, i_e, l_{fv}, k_{ev}) \prod_v \{nj\}_v(l_{fv}, k_{ev})$$

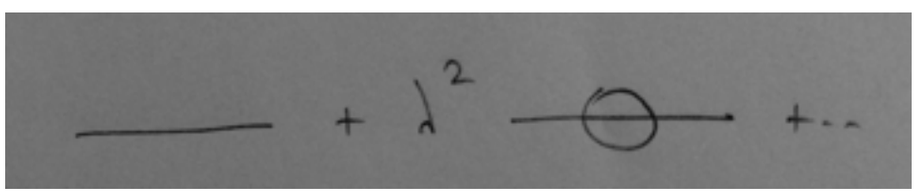


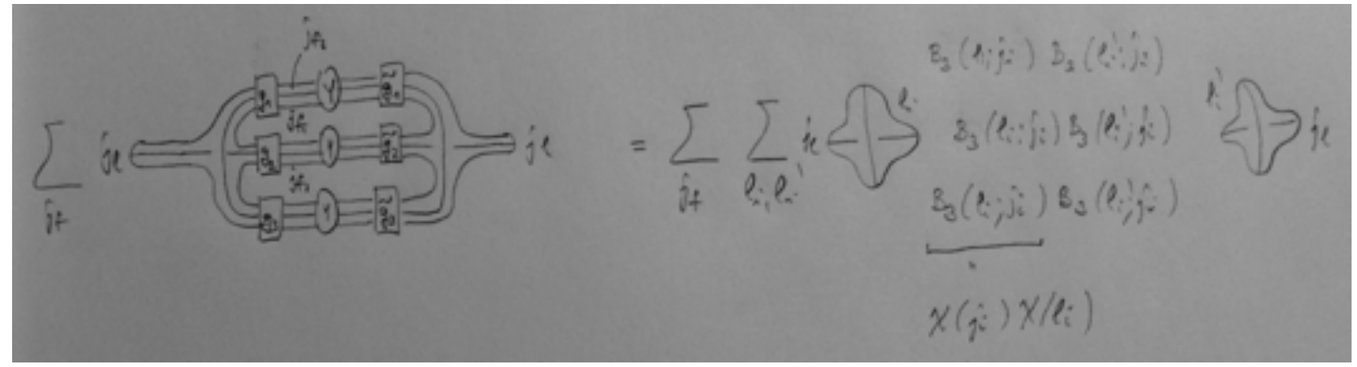
Using results in the literature by Naimark, Kerimov and Verdier (which required a careful fixing of conventions), *the boost integral in the edge amplitude can be done explicitly in terms of* ***finite*** *summations of Γ functions*

- **Pure state sum model formulation also for Lorentzian signature**
- **Improved computing power: complexity now comparable to Euclidean models, use of RR possible**

(Anderson et al. '70)

<u>Example: toy self-energy (4-valent vertices)</u>





Schematically:

$$\sum_{j_f}^{\Lambda} \prod_f (2j_f+1)^\alpha \chi(j_i)^2 \chi(j_i)^2 \chi(j_i)^2 \Big(\underbrace{\sum_{l_f} \chi(l_i)\chi(l_i)\chi(l_i)\{6j\}}_{\text{Computed using RR, rapidly convergent}} \Big)^2 \sim o(\Lambda^{3(\alpha-1)})$$

Logarithmic divergence (cfr. Riello '13)

For the true self-energy (5-valent vertices) we need to handle integrals over virtual irreps
(wip with Pietro Donà and Marco Fanizza)

Plenty of applications

Scaling behaviour:

- For the simplified model, $Z^{EPRLs}_{\mathcal{C}_4} \sim N^{F-3E-3/2\sum_v(n_v-1)} = N^{\chi-5E+V/2}$
- For the EPRL model, work in progress to estimate contribution of additional l summations

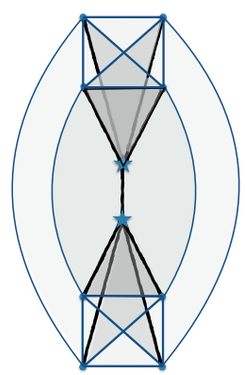
Self-energy and divergences

(wip Marco Fanizza and Pietro Donà)

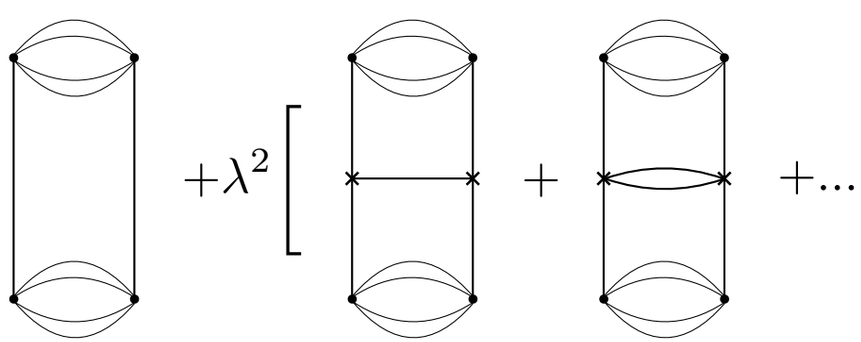
Calculations of physical amplitudes

- Black hole tunnelling

  (Rovelli, wip Marios Christodoulou and Fabio D'Ambrosio)

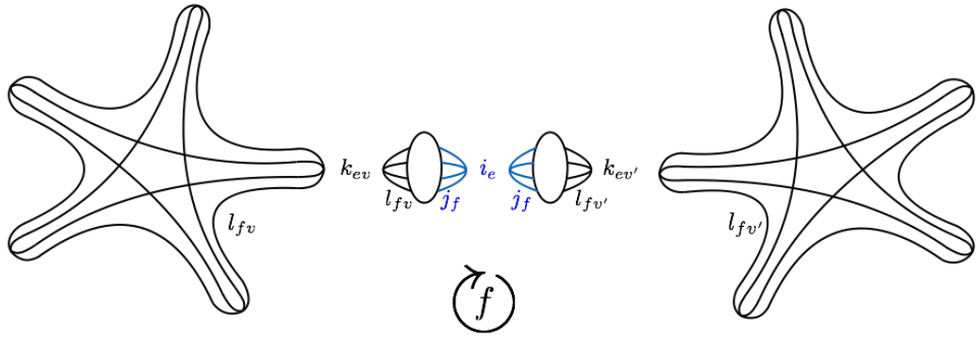


- Cosmological dipole models (Rovelli Bianchi Vidotto) (wip Gabriele Stagno and Giorgio Sarno)



Conclusions

$$Z_{\mathcal{C}} = \sum_{j_f, i_e, l_{fv}, k_{ev}} \prod_f (2j_f + 1) \prod_e A_e^\gamma(j_f, i_e, l_{fv}, k_{ev}) \prod_v \{nj\}_v(l_{fv}, k_{ev})$$



1. Computational complexity of Lorentzian EPRL model under better control

Vertex weights purely SU(2); edge weights can be computed numerically or investigated analytically:

- as integrals over r
- as summations of Γ functions
- using recursion relations

- Pure state sum model formulation also for Lorentzian signature
- Improved computing power: still very hard, but complexity now comparable to Euclidean models
- Explicit calculations in progress, self-energy and divergences, BH tunnelling, spin foam cosmology

2. New perspectives on semiclassical limit

- New analytic approach to asymptotics on arbitrary triangulations $\longrightarrow$ curvature?
- Extendable to spin foams not dual to triangulations $\longrightarrow$ what asymptotic formula?

Geometric picture in this form? Barrett's large spin limit must be re-understood

- Vertex weights give Euclidean Regge action
- Edge weights boost tetrahedra in going from one vertex to the next
- Plaquette action from edge weights: what geometric meaning?

3. Possible formal developments

Pachner moves and renormalisation? (see Dittrich and coll., Freidel and coll., Livine, Dupuis, Bonzom and coll.)
Asymptotic via RR? Inclusion of Λ? (Han and coll., Livine and coll.)
New perspectives on Wick rotation? On γ=i? (Noui, Perez)