Numerical Methods in Spin Foams

Recent developments and Outlook

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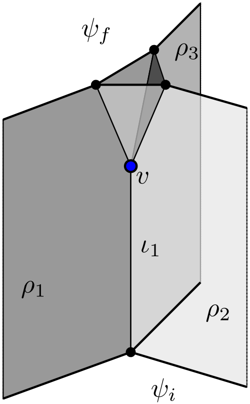
Spin foam gravity [Rovelli, Reisenberger, Barrett, Crane, Freidel, Livine, Krasnov, Perez, Speziale, Engle, Pereira, Kaminski...]



- **Non-perturbative path integral of geometries**
- Regulator: **Discretization** / 2-complex
- **Quantum geometric** building blocks
  - (Constrained) **topological quantum field theory**
  - **Discrete** area spectrum
- Physical content: **Transition amplitudes**
  - **Single building block** $\sim$ **discrete gravity** [Conrady, Freidel '08, Barrett, Dowdall, Fairbairn, Gomes, Hellmann '09, Kaminski, Kisielowski, Sahlmann '17, Liu, Han '18, Simão, S.St. '21]
  - Quantum amplitudes (not Wick-rotated)

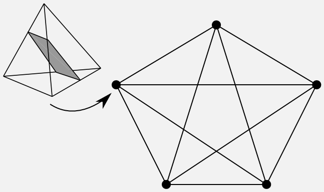
Derived from **general relativity**

No reference to background geometry

Aim to implement **diffeomorphism symmetry**

Computational challenges

$$Z(\Delta^*) = \sum_{\rho_f, \iota_e} \prod_{f \in \Delta^*} \mathcal{A}_f(\rho_f) \prod_{e \in \Delta^*} \mathcal{A}_e(\iota_e) \prod_{v \in \Delta^*}$$



- **Explicit** calculation of **vertex amplitude** (amplitude of a 4-simplex)
- **Sum over representations** and intertwiners

Non-perturbative calculations often require a lot of **numerical resources**.

Progress frequently comes from **analytical understanding**.

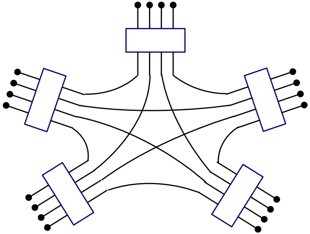
Recent (numerical) progress

- Resources: **Handbook of Quantum Gravity** articles [Dona, Han, Liu '22, Asante, Dittrich, S.St. '22]
- Explicit calculation of **vertex amplitude**:
  - `sl2cfoam` for EPRL/FK model [Donà, Fanizza, Sarno, Speziale '19, Gozzini '21]
  - $\mathrm{SU}(2)$ BF theory [Donà, Fanizza, Sarno, Speziale, '17, Asante, Simão, S.St. w.i.p.]
- Utilizing **semi-classical** insights
  - **Effective spin foams** [Asanta, Dittrich, Haggard PRL '20, Asante, Dittrich, Padua-Argüelles '21]
  - Acceleration operators for **series convergence** [Dittrich, Padua-Argüelles '23]
  - **Hybrid representation** of spin foams [Asante, Simão, S.St. '22]
  - Complex critical points [Han, Huang, Liu, Qu '21, Han, Liu, Qu '23]
  - Restricted spin foam models [Bahr, S.St. '15, Bahr, Rabuffo, Klöser '16, Assanioussi, Bahr '20]
- **Monte Carlo methods**
  - Markov chain Monte Carlo on Lefshetz thimbles [Han, Huang, Liu, Qu, Wan '20]
  - Random sampling Lorentzian EPRL model [Donà, Frisoni '23]

Outline

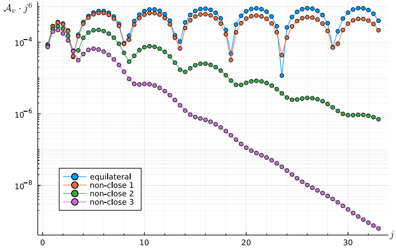
Semi-classical properties of spin foams

- **Critical points** dominate vertex amplitude for large representations [Conrady, Freidel '08, Barrett, Dowdall, Fairbarn, Gomes, Hellmann '09, '10, Kaminski, Kisielowski, Sahlmann '17, Han, Liu '18, Simão, S.St. '21]
  - **Coherent** boundary data [Livine, Speziale '07]



Critical points of spin foam vertex amplitude

- **Geometric 4-simplices**
  - Oscillate with **Regge action**
- Vector geometries (degenerate)

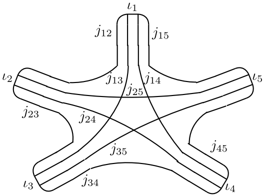


- **Exponential suppression** away from critical points

How can we evaluate vertex amplitudes **explicitly**?

Explicit evaluation of vertex amplitudes I

- Example 1: **SU**(2) $\{15\}j$**-symbol** (BF vertex amplitude) [Dona, Fanizza, Sarno, Speziale '17, Dona, Gozzini, Sarno '20]
  - 4-simplex for ON spin network basis
  - Auxiliary variable x: finite sum, constrained by intertwiner and spin labels

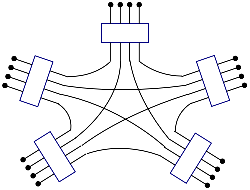


$$= (-1)^{\sum_i j_i + \sum_i \iota_i} \sum_x (2x+1) \begin{Bmatrix} \iota_1 & j_{25} & x \\ \iota_5 & j_{14} & j_{15} \end{Bmatrix} \begin{Bmatrix} j_{14} & \iota_5 & x \\ j_{35} & \iota_4 & j_{45} \end{Bmatrix}$$

$$\times \begin{Bmatrix} \iota_4 & j_{35} & x \\ \iota_3 & j_{24} & j_{34} \end{Bmatrix} \begin{Bmatrix} j_{24} & \iota_3 & x \\ j_{13} & \iota_2 & j_{23} \end{Bmatrix} \begin{Bmatrix} \iota_2 & j_{13} & x \\ \iota_1 & j_{25} & j_{12} \end{Bmatrix}$$

Explicit evaluation of vertex amplitudes II

- Example 2: **Coherent SU(2) BF amplitude** [Dona, Fanizza, Sarno, Speziale '17]
  - **Highly oscillatory** 12-dim integration (4 SU(2))



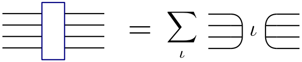
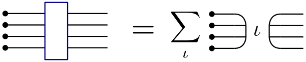
$$= \int_{\mathrm{SU}(2)} \prod_a dg_a \prod_{a<b} \left\langle j_{ab}, -\vec{n}_{ba} \left| D^{j_{ab}}(g_b^{-1}) D^{j_{ab}}(g_a) \right| j_{ab}, \vec{n}_{ab} \right\rangle$$

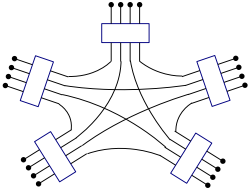
$$= \int_{\mathrm{SU}(2)} \prod_a dg_a \prod_{a<b} \left(\left\langle \tfrac{1}{2}, -\vec{n}_{ba} \left| g_b^{-1} g_a \right| \tfrac{1}{2}, \vec{n}_{ab} \right\rangle \right)^{2j_{ab}}$$

Explicit **numerical integration** converges slowly.

Ideal setting to apply **stationary phase approximation**.

Explicit evaluation of vertex amplitudes II

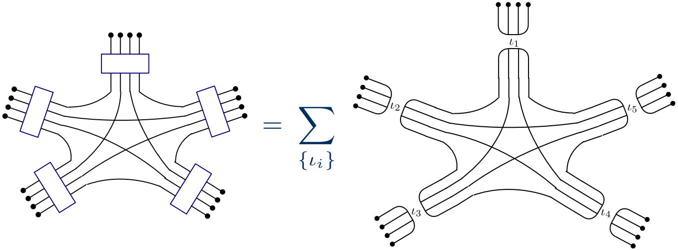
- Example 2: Coherent SU(2) BF amplitude [Dona, Fanizza, Sarno, Speziale '17]
- **Perform integrations analytically**: Haar projector 
  - Expand coherent intertwiner in ONB 



$$= \int_{\mathrm{SU}(2)} \prod_a dg_a \prod_{a<b} \left(\langle \tfrac{1}{2}, -\vec{n}_{ba} \,|\, g_b^{-1} g_a \,|\, \tfrac{1}{2}, \vec{n}_{ab} \rangle \right)^{2j_{ab}}$$

Explicit evaluation of vertex amplitudes II

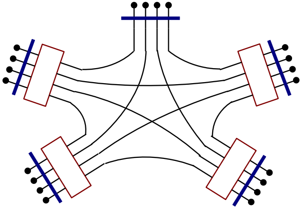
- Example 2: Coherent $\mathrm{SU}(2)$ BF amplitude [Dona, Fanizza, Sarno, Speziale '17]
- **Intertwiner range** grows with increasing spin labels
  - Optimize by using **tensor network methods** [Gozzini '21]



Traded **group integrations** for finite (potentially large) **sum over ON intertwiners**.

Explicit evaluation of vertex amplitudes III

- Example 3: **Lorentzian EPRL SL$(2,\mathbb{C})$ amplitude** [Dona, Fanizza, Sarno, Speziale '19, Gozzini '21]
  - Highly oscillatory 24-dim integration (4 SL$(2,\mathbb{C})$)
  - Blue bar: Y_γ map specifies SL$(2,\mathbb{C})$ irreps

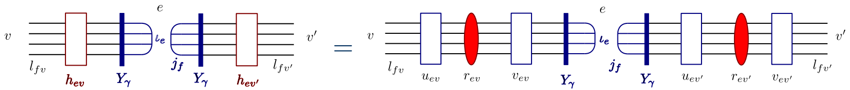


$$= \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=2}^{5} dh_a \prod_{a<b} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}, -\vec{n}_{ba}, j_{ab}, \vec{n}_{ab}} (h_b^{-1} h_a)$$

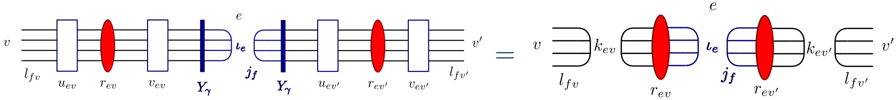
More challenging, but also much more interesting than the case for SU(2) BF...

Explicit evaluation of vertex amplitudes III

- Example 3: Lorentzian EPRL $\mathrm{SL}(2,\mathbb{C})$ amplitude [Dona, Fanizza, Sarno, Speziale '19, Gozzini '21]
- **Key idea by Speziale** [Speziale '16]: **Decompose** $\mathrm{SL}(2,\mathbb{C})$ integration
  - $D^{(\gamma j,j)}_{j,m,j,m'}(h_b^{-1}h_a) = \sum_{l\geq j}\sum_{n=-l}^{l} D^{(\gamma j,j)}_{j,m,l,n}(h_b^{-1})D^{(\gamma j,j)}_{l,n,j,m'}(h_a)$
  - Cartan decomposition of $\mathrm{SL}(2,\mathbb{C})$: $h = u\, e^{\frac{r}{2}\sigma_3}\, v^{-1}$ for $u, v \in \mathrm{SU}(2)$ and $r \in [0,\infty)$.

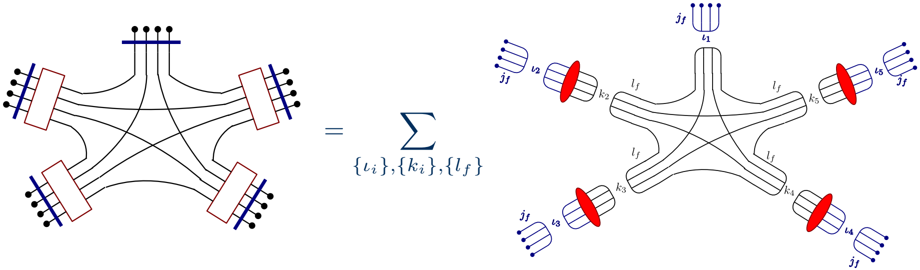


- After $\mathrm{SU}(2)$ integrations: four **1d boost integrals** remain!



Explicit evaluation of vertex amplitudes III

- Example 3: Lorentzian EPRL SL$(2, \mathbb{C})$ amplitude [Dona, Fanizza, Sarno, Speziale '19, Gozzini '21]
  - **Sums** over **auxiliary spins** l_f and **intertwiners** k_i.
  - **Shelled sum** over l_f [Dona, Fanizza, Sarno, Speziale '19]: from j_f to $j_f + \Delta j$



Insights and optimizations culminated in the package `sl2cfoam-next`.

Lorentzian quantum regime of **several simplices accessible** (at high costs).

Related results and methods

- **Verification** of **asymptotic formula** in Lorentzian EPRL model [Gozzini '21]
- **Triangulations with several 4-simplices** / vertices
  - Numerical confirmation of **flatness problem** [Dona, Gozzini, Sarno '20, Gozzini '21, Dona, Frisoni '23]
  - **Divergence behavior** of spin foams [Dona '18, Dona, Frisoni, Wilson-Ewing '22, Dona, Frisoni '23]
  - Search for critical points [Dona, Gozzini, Sarno '19]
  - Contraction of intertwiners as tensor networks [Gozzini '21]
  - Utilizing **Monte Carlo methods** [Frisoni, Gozzini, Vidotto '22, Dona, Frisoni '23]
- **How-to perform calculations of EPRL model** [Dona, Frisoni '22]
- SU(2) BF tools [Asante, Simão, S.St. w.i.p.]
  - 2-complex constructor
  - Different vector geometry parametrization (see also [Dona, Fanizza, Sarno, Speziale '17])
- **Higher valent** (SU(2)) vertex amplitudes [Allen,Girelli, S.St. '22]
- Use quantum amplitudes to **extrapolate to semi-classical regime** [Jercher, S.St., Thürigen '23]

Outline

Developing effective models

- **Larger triangulations** necessary to make contact with **continuum**
  - **Plethora** of configurations to sum over
  - **Not feasible** to perform explicit calculation up to **infinite** spins
- **Physically motivated approximations** are necessary
  - Simpler (to compute) amplitudes
  - **Exclude** (less relevant) configurations
- Possible route: **semi-classical analysis**
  - Semi-classical amplitudes: valid approximation for "large" representations
  - Critical points dominate for "large" representations

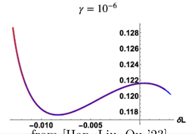
Both ideas have potential to **accelerate numerical calculations**.

When are such **approximations justified**?

Important recent developments

Complex critical points [Han, Huang, Liu, Qu '21, Han, Liu, Qu '23]

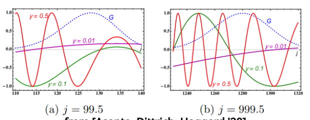
- **Asymptotic analysis** of spin foams on **larger complexes**
  - **More** than **real critical points** contribute (at finite scales).
  - In particular **non-flat** configurations
  - Stronger suppression with larger deficit angle



from [Han, Liu, Qu '23]

Effective spin foams [Asante, Dittrich, Haggard '20, '21, Asante, Dittrich, Padua-Arguelles '21, '22, Dittrich, Padua-Arguelles '23]

- **Area Regge** path integral + **gluing constraints**
  - **Discrete area spectrum**
  - **Replace** full amplitude by exponentiated area Regge action
  - Gluing constraints peaked on **shape matching**



(a) $j = 99.5$ (b) $j = 999.5$
from [Asante, Dittrich, Haggard '20]

Length Regge emerges for small Immirzi parameter γ.

Area vs. length Regge calculus

- Consider **Regge action** [Regge '61] **in 4d**
  - A_t: area of triangle t
  - ϵ_t: deficit angle
  - Θ_t: exterior curvature angle

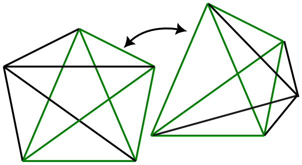
$$S_{\text{Regge}} = \sum_{t \in \text{bulk}} A_t \, \epsilon_t + \sum_{t \in \text{bdry}} A_t \, \Theta_t$$

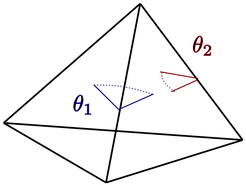
- **Relation** length and area Regge calculus [Barrett, Roczek, Williams '97, Dittrich, Speziale '07, Bahr, Dittrich '09, Neiman '13, Asante, Dittrich, Haggard '18]
  - Length Regge: $A_t(l_e)$ and $\epsilon_t(l_e)$
  - Area Regge: A_t and $\epsilon_t(A_t)$
- Area Regge enforces $\epsilon_t = 0$, i.e. **flatness**
  - Schläfli identity: $\sum_{t \in \sigma} A_t \, \delta\theta_t = 0$.

Simplicity constraints should reduce area to length Regge.

Area Regge + gluing constraints [Asante, Dittrich, Haggard '20]

- **Areas** are **fundamental variables**
- 4-simplex **not determined** by 10 areas [Asante, Dittrich, Haggard '18]
  - Multiple lengths configurations for same areas possible
- Take **two 4-simplices** glued along a common tetrahedron
  - Length Regge: **six edge lengths** must agree
  - Area Regge: four triangle areas must agree
  - Typically **more triangles** than edges
- Metric **discontinuous** / torsion degrees of freedom
- Gluing constraints impose **simplicity weakly**
  - Gaussian peaked on **matching angles** (shape matching)
- **Competing effects**: Gaussian vs. oscillating amplitude
  - γ controls frequency of oscillations

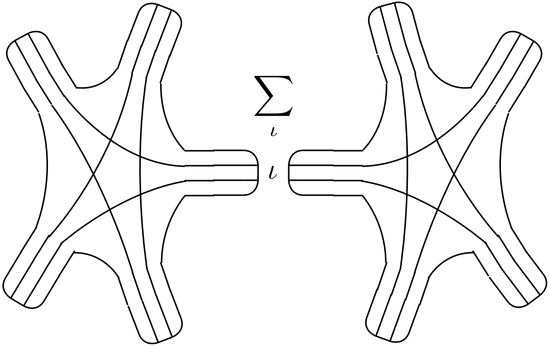




Can we derive a **similar representation** for **full spin foams**?

Spin foam as coherent amplitudes

- Equip each vertex with **independent set of coherent data**



Spin foam as coherent amplitudes

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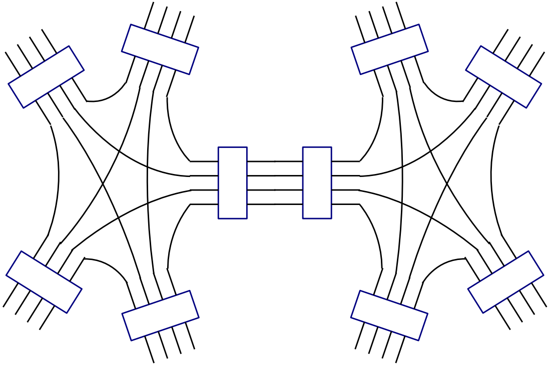

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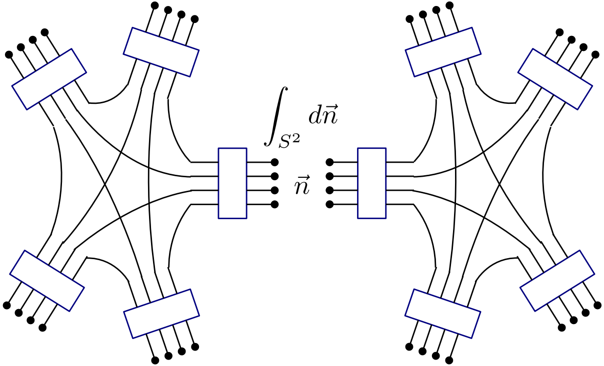

Spin foam as coherent amplitudes

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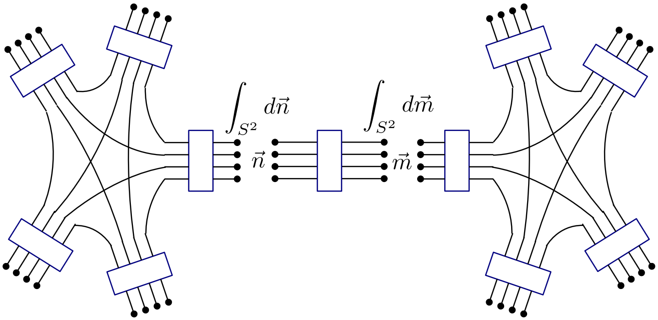
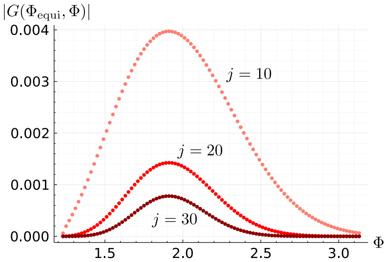


Interpolate between vertices by **gluing constraints**.

Hybrid representation of spin foams [Asante, Simão, S.St. '22]

Properties of gluing constraints [Asante, Simão, S.St. '22]

- **Gluing constraints** peaked on **matching tetrahedra**
  - Analytic formula away from critical points



Suggest an **intermediate regime** in spin foams

Non-matching, semi-classical vertices

interpolated by gluing constraints

- Expect **major contribution** from **matching critical points**
  - Useful for "guessing" relevant configurations

Can we use this **optimize calculations** in the full theory?

What about the coherent state integrations?

Hybrid algorithm(?) [Asante, Simão, S.St. '22]

- Idea: **approximate** full amplitude by **asymptotic formulae** for each vertex
  - Each vertex has its **independent set** of coherent data
  - Exchange order of integrations
- Challenge: **remaining intergrations** over coherent data
  - Hope: **main contribution from (matching) critical points** of vertices
  - Replace integration by sum over critical points
- Replacement too simple: **wrong scaling** behavior of amplitudes
  - Hessian for larger triangulations is **non-local** [Dittrich, S.St. '11, Dittrich, Kaminski, S.St. '14, Banburski, Chen, Freidel, Hnybida '14]
  - Some coherent state integrals are gauge
  - Vertex amplitude away from critical points?

Valuable insight into what type of configurations **might be relevant** at larger spins.

Build new effective spin foam model (including vector geometries?)

Alternative route: complex critical point algorithm [Han, Huang, Liu, Qu '21, Han, Liu, Qu '23]

Related results

- **Area variables and area metrics**
  - Algorithm from area to length configurations [Asante, Dittrich, Haggard '18, Asante, w.i.p.]
  - Continuum limit of linearized effective spin foams [Dittrich '21]
  - Area metrics and continuum theories [Dittrich, Kogios '22, Borissova, Dittrich '22]
  - Twisted geometries and area variables [Dittrich, Padua-Arguelles '23]
  - Lorentzian configurations and irregular causal structures [Asante, Dittrich, Padua-Arguelles '21, Dittrich, Gielen, Schander '21, [Dittrich, Asante, Padua-Arguelles '21] [Dittrich, Padua-Arguelles '23]
- **Saddle point finder** in spin foams [Huang, Huang, Wan '22]
- **Acceleration operators** for convergence [Dittrich, Padua-Arguelles '23]

Fast computability of amplitudes chance to explore **larger triangulations**

How to sum over **thousands of variables**?

Is **convergence** fast (enough)?

Outline

What is importance sampling?

- Lattice theories: **importance sampling** used to **approximate non-perturbative integrals**
- Example: scalar lattice field theory

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_i d\phi_i \; \mathcal{O}(\{\phi_i\}) \; e^{-S[\{\phi_i\}]} \approx \frac{1}{n} \sum_{k=1}^{n} \mathcal{O}(\{\phi_i\}_k)$$

- The probability distribution $\frac{e^{-S[\phi]}}{Z}$ is used to **sample configurations**
  - Random walk through configuration space
  - **Numerical costs** grow with $\sqrt{n}$, not system size
- Monte Carlo algorithms are **more intricate** than it seems
  - Proposal scheme: must be **ergodic** and satisfy Detailed Balance
  - **Explore large configuration space**: tune acceptance rate

Importance sampling can be **highly efficient** in studying systems with many degrees of freedom - if the results **converge** quickly!

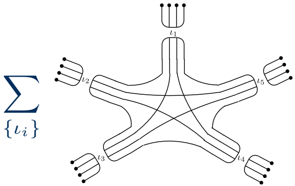
Numerical sign problem in spin foams

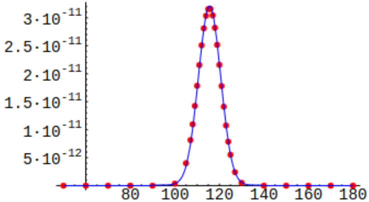
- **Sign problem**: Spin foam amplitudes (as is) **cannot** be used to **define importance sampling**
  - They do not define a probability distribution
  - Exception are highly symmetric setups [Bahr, S.St., Bahr, Klöser, Rabuffo '17]
- Instead, we can **modify** the system
- Monte Carlo on a **Lefshetz thimble** [Han, Huang, Liu, Qu, Wan '20]
  - Modify integration contour such that **imaginary part is constant**
- Choose / define a **new probability distribution**
  - Constant distribution: Random sampling [Dona, Frisoni '23]
  - Reweighting procedure: sample w.r.t. absolute value and absorb phase in observable
- **Guess** a **suitable probability distribution** for system

Still, **sign problem** might be to **severe: no convergence**.

Importance sampling for single vertex [S.St. w.i.p.]

- Recall **coherent SU(2) BF amplitude** from beginning
  - $\{15\}$ symbol contracted with coherent intertwiners.
- Absolute value of ι is **nicely peaked** [Livine, Speziale '07]





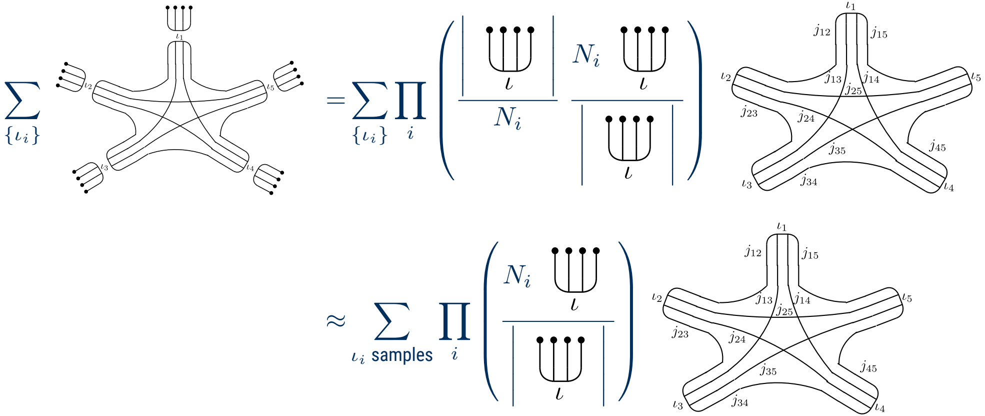
from [Livine, Speziale '07]

Use **boundary data** to **sample** relevant intertwiner labels.

To do so, we must **modify the amplitude**.

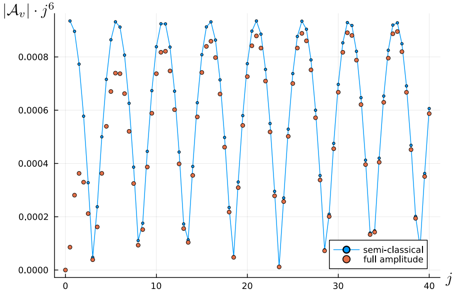
Importance sampling for a single vertex II

- Expand amplitude as follows:



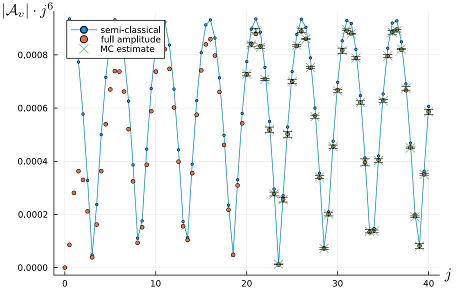
Test: SU(2) coherent vertex amplitude

- Coherent vertex amplitude: **importance sampling from boundary data**
  - Equilateral boundary data: **sum over** $(2j+1)^5$ **combinations**
  - 10 runs with 10^6 **samples** for $j = 20, \ldots, 80$.



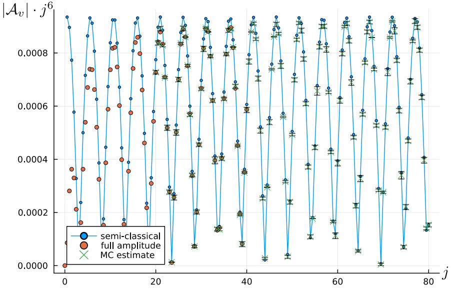
Test: $\mathrm{SU}(2)$ coherent vertex amplitude

- Coherent vertex amplitude: **importance sampling from boundary data**
  - Equilateral boundary data: **sum over** $(2j+1)^5$ **combinations**
  - 10 runs with 10^6 **samples** for $j = 20, \ldots, 80$.



Test: $\mathrm{SU}(2)$ coherent vertex amplitude

- Coherent vertex amplitude: **importance sampling from boundary data**
  - Equilateral boundary data: **sum over** $(2j+1)^5$ **combinations**
  - 10 runs with 10^6 **samples** for $j = 20, \dots, 80$.



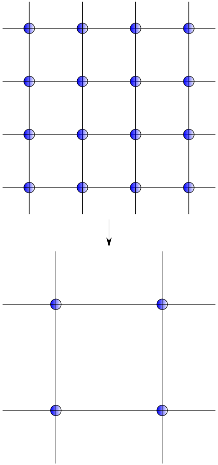
Good agreement and fast convergence!

Sign problem present: **global phase** of amplitude

Outline

Tensor Network Renormalization [Levin, Nave '07, Gu, Wen '09]

- Write system as a **tensor network**
  - Tensor: **local amplitude** with bdry finite Hilbert space
  - Network: **contract** shared indices between tensors
- **Locally coarse grain** tensor network
  - Use singular value decomposition (SVD) to **truncate**
  - **Explicitly sum** over degrees of freedom
- Works well for oscillating amplitudes
- Various algorithms / formulations exist
- **Challenge**: casting system into tensor network form



Opposite to Monte Carlo: evaluate partition function in parts

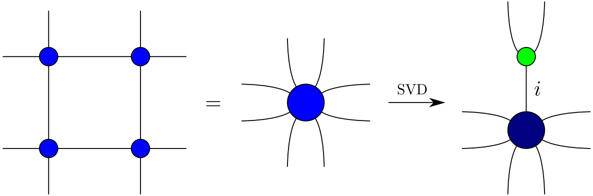
No sampling, not affected by sign problem.

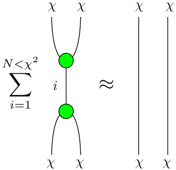
2D example of the algorithm

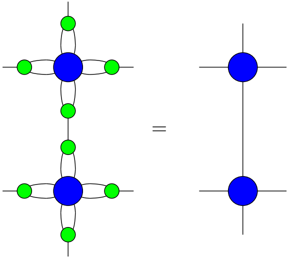
- **Effective tensor** after contraction
  - Bdry data grows **exponentially**
- **Truncation** necessary
  - Control error?
- **Variable trafo + truncation** from SVD
  - Rewrite tensor as matrix

$$M_{AB} = \sum_{i=1}^{\chi^2} U_{Ai}\, \lambda_i\, (V^\dagger)_{iB}$$

- Singular values indicate relevance
  - $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\chi^2} \geq 0$

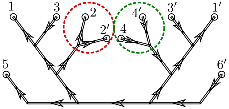


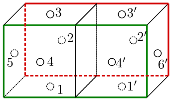


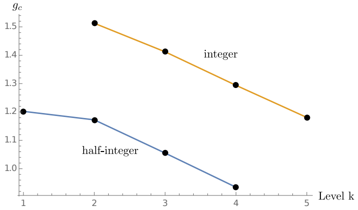


Study **flow of tensors**

Fusion basis in 3d





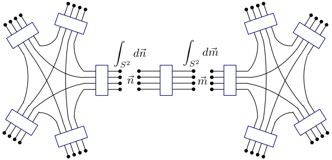


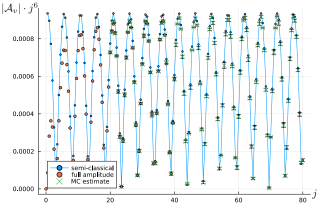
- **Spin foams** in tensor network language **challenging**
  - Decorated tensor networks [Dittrich, Mizera, S.St. '14, Delcamp, Dittrich '16]
- **2d Analogue models** [Dittrich, Eckert, Martin-Benitp '11, Dittrich, Eckert '11, Dittrich, Martin-Benito, Schnetter '13, Dittrich, Martin-Benito, S.St. '13, Cameron, Dittrich, Schnetter, S.St. '16]
- 3d: lattice gauge theory as **fusion basis** [Dittrich, Geiller '16, Delcamp, Dittrich, Riello '16]
  - Excitations (curvature and torsion) located at punctures
- **Fusion basis algorithm** for SU$(2)_k$ LGT [Cunningham, Dittrich, S.St. '20]
  - Coarse graining of punctures
  - Determine **phase structure**
  - Compute **observables** (Wilson loops)
- **Interesting works**·
  - 2d Lorentzian quantum Regge calculus [Ito, Kodah, Sato '22]
  - 2d ϕ^4 lattice field theory [Delcamp, Tilloy '20]

Outline

Summary and Outlook

- **Overview** of **numerical methods and progress** in spin foams
- **Explicit computation** of vertex amplitudes
  - Lorentzian quantum regime with `sl2cfoam-next`
- **Effective models** w/ semi-classical methods
  - Large triangulations possible with semi-classical approximations
  - Emergence of **length Regge**
- **Unknown potential** of Monte Carlo
  - Use semi-classics to guess prob. distribution
  - Test of **coherent vertex amplitude** encouraging!
- Other **promising technologies**
  - **Quantum computing** [Zhang, Huang, Song, Guo, Song, Dong, Wang, Hekang, Han, Wang, Wan '20, Mielczarek '18]
  - **Deep Machine Learning**

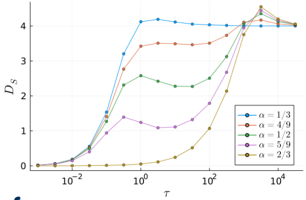


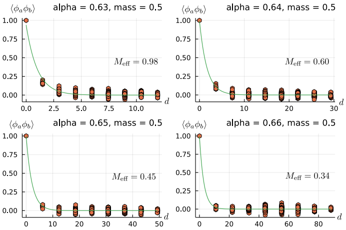


Open issue: how to **combine quantum and semi-classical regime**?

Eventual goal: observables

- Spin foam **Spectral dimension** [S.St., Thürigen '18, Jercher, S.St., Thürigen '23]
  - **Effective dimension** of quantum space-time from (spectrum of) Laplace operator
  - Depends on scaling behavior of spin foam amplitudes





- **Matter and spin foams** [Ali, S.St. '22]
  - **Matter-gravity observables** in spin foams?
  - Ex.: Correlation length of massive scalar field coupled to cuboid spin foams
- **Yang-Mills** coupled to cuboid spin foams [Asante, S.St. w.i.p.]
- **Relational dynamics** in Lorentzian spin foams [Jercher, S.St. w.i.p.]

Thank you for your attention!