# Quantum dynamics of LQG

## Hypersurf. Def. Alg., Q'um Non-Deg., Anomalies, Dens. Weigths, Renormalisation

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ILQGS, 11.10.2022

arXiv: 2207.08299 (strategy); 2207.08302, 2207.08290 (models); 2207.08294, 2207.08291 (tools)

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# Classical Hypersurface Deformation Algebra

- Let $Q := \sqrt{\det(q)}$, $D$ dens. w. one spat. diff. const., $C_w$ dens. w. $w$ Ham. const.
- Classical hypersurface algebra $\mathfrak{h}$ [Hojman, Kuchar, Teitelboim]

$$\{C_w[M],\ C_w[N]\} = -D[Q^{2(w-1)}\ q^{-1}\ (M\ dN - N\ dM)]$$

- Observations:
  - Ill-defined for degenerate metrics ($Q = 0$) unless $w \geq 2$
  - Trivial (Abelian) if integrand of $D$ has Lebesgue measure zero support

# Representations and density weights

- Every single term in $C_w$ (vacuum, cosm. const., matter) couples to $E^a_j$
- LQG: in order that $C_w[N]$ be densely defined, pick vacuum s.t. $E^a_j \, \Omega = 0$
- $\Rightarrow$ quantum degenerate vacuum $Q \, \Omega = 0$
- Proposition: This already fixes a rep. of Narnhofer-Thirring type (e.g. AL rep.)
- Proposition: If $C_w$ dep. quadratically on $A^j_a$ and vol. op. (needed for Lorentzian Ham. constr., cosm. const., matter terms) densely defined then a simplicial regularisation (Riemann sum over tetrahedral cells of coordinate volume $\epsilon^d$) of $C_w$ is densely defined on $\mathcal{H}$ iff
  - $w = 1$
  - $A, E$ smeared in $1, d-1$ dimensions respectively

# The Tension

- For $w = 1$: classical hypersurf. alg. $\mathfrak{h}$ well-defined iff $Q > 0$
- LQG rep.: SNWF excitations $\mathcal{D}$ on graphs with finite number of vertices
- all SNWF quantum degenerate (zero volume Lebesgue a.e.)
- $\epsilon \to 0$ limit: delicate as naive $\epsilon \to 0$ limit divergent (naively $Q^{-1} \equiv \infty$)
- inverse $Q$ powers must annhilate SNWF a.e. (Tychonov def., P.B. id. [TT], ...)
- Proposals:
    - A: rep.of $\mathfrak{h}$ on $\mathcal{D}$:
      conv. in operator topology eploiting diffeo inv. [Rovelli, Smolin, TT]
    - B: rep. of $\mathfrak{h}$ on subsp. of distrib. dual $\mathcal{D}^*$ ("habitat") [Gambini, Lewandowski, Marolf, Pullin]
- Anomalies:
    - A: closes with non-trivial but wrong q'um structure functions (volume kills new vertices)
    - B: closes with trivial i.e. wrong q'um structure functions

# Origin of the Tension

- Technical reason: $D[Q^{2(w-1)}\ q^{-1}(M\ dN - N\ dM)]$ is
  - Classically: Riemann sum of $N_\epsilon = \epsilon^{-d}$ terms (compact spat. top.) of size $\epsilon^d$
  - Q'um: sum of $N_\gamma$ =const. terms (no. of vertices) of size $\epsilon$ due to discrete derivative $M(v)\ N(v+\epsilon) - M(v+\epsilon)\ N(v)$
  - $\epsilon^d \times \epsilon^{-d} \to O(1)$ but $\epsilon^1 \times N_\gamma \to 0$
- Perspectives:
  - I: non-deg. LQG vacuum $\Omega_0$ with condensate $<\Omega_0,\ Q\,\Omega_0>= Q_0 > 0$ [Koslowski, Sahlmann]
  - II: non-standard dens. w. $2 > w > 1$ : match s.t. in reg. comutator on habitat, $\epsilon^{-1}$ multiplies discrete der. [Varadarajan et. al.]
- Reservations:
  - I: $\Omega_0$ not in domain of reg. Ham. constr.; excitations still suffer from $N_\gamma$ =const. while $N_\epsilon \to \infty$ needed.
  - II: $w \neq 1$, "electric shift" strategy [Ashtekar, Varadarajan] presently geared to Euclidian vacuum QG (Lorentzian vacuum: Wick transform? [TT; Varadarajan])

# Summary

- The natural density weight is $w = 1$
- For $w = 1$ classical $\mathfrak{h}$ requires non-degenracy
- LQG SNWF q'um degenerate a.e.
- Reason why q'um representation of $\mathfrak{h}$ meets severe difficulties
- q'um rep. violates necessary assumption about very definition of $\mathfrak{h}$
- Strategy: Find new rep. which is q'um non-degenerate
- make q'um non-degeneracy part of of definition of anomaly freeness
- Further plan of talk:
    - Proof of principle: exact, anomaly-free, q'um non.deg. q'ion of Smolin's $U(1)^3$ model q'um integrability
    - Renormalisation: systematic construction of q'um non-deg. rep.

# Definition of classical $U(1)^3$ model

- Hamiltonian definition [Smolin]:
  Take Euclid. vac. GR in Ashtekar-Barbero variables, drop $A^2$ terms from $C_1[M]$
- Lagrangian definition [Bakhoda, TT]:
  Take Euclid. vac. GR in self-dual variables, drop $A^2$ terms from $L$
- Almost Euclidian vacuum GR, but Abelian structure group
- Classical hypersurf. def. alg. $\mathfrak{h}$ unchanged
- in particular: still non-trivial, non-polynomial struct. fns.
- ideal test laboratory for many technical/conceptual issues of QG [Varadarajan et al] both canonical and covariant

# Definition of q'um non-deg. $U(1)^3$ QG

- Narnhofer-Thirring type of rep.

$$< \Omega, w[F]\Omega >= \delta_{F,0},\ w[F] = \exp(-i\ F[A]),\ E[f]\Omega = 0,\ F[A] := \int d^3x\ F^a_j\ A^j_a$$

- $F$ : form factor, generalised "holonomies" $w[F]$ discont., "fluxes" $E[f]$ cont.
- Geometrical ops. diagonal, e.g. volume

$$V(R)\ w[F]\ \Omega = \ell_P^3\ [\int_R d^3x\ \sqrt{|\det(F)|}]\ w[F]\ \Omega$$

- q'um non-deg dense domain: $\det(F) \neq 0$
- solution of Gauss constraint: $\partial_a F^a_j = 0$
- spatial diffeo $D[u]$, Ham. constr. $C_w[M]$: ill-defined as $A \not\exists$
- No rep. of $\mathfrak{h}$ on $\mathcal{H}$. But: can exponentiate $\mathfrak{H} := \exp(\mathfrak{h})$ on $\mathcal{H}$
- $U^w(u,M)\ w[F]\ \Omega := \exp(D[u] + C_w[M])\ w[F]\ \Omega = w[(e^{X^w_{u,M}} \cdot K)(0,F)]\ \Omega$
- $X^w_{u,m}$: HVF of $D[u] + C_w[M]$, $K(G,F) := F$ momentum coordinate fn.

# Properties of $U(1)^3$ QG

- to best of knowledge: first q'um realisation of Bergann-Komar "group"
- derived using standard simplicial reg. of LQG, polymerisation
- works for any $w$, in part. natural weight $w = 1$
- $U^w(u, M)$ densely defined, in fact unitary, reduces to spatial diffeo $\varphi^u_{t=1}$ for $M = 0$
- implemented w/o regulator directly on $\mathcal{H}$, no habitats necessary
- anomaly freeness realised: q'um algebra encoded by Hamiltonian flow of classical constraints on non-deg. form factors
- q'um non-degeneracy crucial: HVF otherwise ill-defined

# Properties of $U(1)^3$ QG

- implementation of electric shift/gauge covariant diffeo perspective [Giesel, TT 06; Ashtekar, Varadarajan 21] to all orders in Abelian context
- Ham. flow $[e^{X^w(u,M)} \cdot K](0,F)$ computable at N-th order wrt $u, M$: while linear in $F$ for $M=0$, e.g. for $w=2$ nested polynomial of order $N+1$ in F depending on spat. der. of order N
- Ham. constr. action: Mollify CNW-FF $F^a_j(x) = \sum_e \; n^j_e \; \int_e \; dy^a \; \delta(x,y)$, then:
  1. action along whole graph (not only vertices), no abrupt loop attachment,
  2. action on charges non-polynomial
- Using habitats anyway, access to $\mathfrak{h}$ rather than $\mathfrak{H}$: anomaly free by construction
- perfect match: group averaging vs. red. phase sp. q'ion (relational observables)
- Physical HS and Hamiltonian: non-linear, self-interacting electrodynamics: N-point Wightman fns. not determined by 2-pt fn.
- non-relational weak Dirac observables of CDJ type [Capovilla, Dell, Jacobson]
- Spin foam derivation: Discrete/Bohr measures rather than formal Lebesgue, simplicity constraint from first principles, Abelian SFM, much simpler!

# Summary of U(1)$^3$ QG

- U(1)$^3$ QG (almost) q'um integrable in Narnhofer-Thirring type of rep.
- Convergence of ideas: canonical, covariant, relational observables, ...
- can be considered paradigm model or "harmonic oscillator" of (L)QG
- highlights the importance to implement q'um non-degeneracy
- LQG techniques otherwise work, density weight unity, no habitats
- Reason for success: HVFs preserve momentum polarisation of phase space
- full (Euclidan) QG: no longer polarisation preserving, more complicated
- New perspective: pert. theory around integrable model = consistent deformation of Euclidian GR [Barbero]
- Non-pert., constructive approach: Hamiltonian Renormalisation

# Generalities of Hamiltonian Renormalisation

- Framework motivated by constructive QFT [Balaban, Glimm, Fröhlich, Jaffe, Osterwalder, Rivasseau, Schrader, Simon, Thirring, ...], in a nutshell:
- $\mathcal{M}$: set of "resolution scales": part. ordered, directed
- family of OS-triples $T_M := (\mathcal{H}_M, \Omega_M, H_M),\ M \in \mathcal{M}$
- given isometric injections: $J_{MM'}\ :\ \mathcal{H}_M \to \mathcal{H}_{M'};\ M \leq M'$ i.e. $J^\dagger_{MM'}\ J_{MM'} = 1_M$ s.t. $J_{M_2 M_3}\ J_{M_1 M_2} = J_{M1 M_3}\ \ \forall\ M_1 \leq M_2 \leq M_3$
- family of OS-triples called consistent iff

$$J^\dagger_{MM'}\ H_{M'}\ J_{MM'} = H_M \quad \forall\ M \leq M'$$

- Then continuum theory $(\mathcal{H}, \Omega, H)$ obtained by inductive limit of HS: $J_M\ :\ \ \mathcal{H}_M \to \mathcal{H}$ and Hamiltonian $H$ s.t.

$$J_{MM'} = J^\dagger_{M'}\ J_M,\ \ H_M = J^\dagger_M\ H\ J_M$$

- Question 1: how to get these structures from given classical theory?
- Question 2: How does it help to find q'um non-degenerate reps.?

# Multi-Resolution-Analysis (MRA) and Wavelets

MRA of wavelet theory = organisational principle of renormalisation

- (Generalised) Multi-Resolution-Analysis (MRA):
  Nested family of sub-HS $V_M,\ M \in \mathcal{M}$ of "1-particle" HS $V = L_2(\sigma)$ of "smearing functions" on spat. slice $\sigma$ s.t.
  i. $V_M \subset V_{M'},\ M \leq M'$
  ii. $\cup_M L_M$ is dense in $L$
  iii. $\cap_M L_M = \mathbb{C}$ (resp. $\{0\}$) for (non-)comp. $\sigma$
  iv. if $M \leq M'$ $\exists$ scale factor $s(M, M') > 0$ s.t. $\forall f \in V_M$ dilatations: $D_{s(M,M')} f \in V_{M'}$
- (Generalised) scaling function $\chi$:
  $\exists$ dimension no. $d(M)$ and fixed, finite set of fns $\chi \in V$ whose rescaled translates $\chi_m^M := D_{d(M)} T^m_{1/d(M)} \chi$ form ONB of $V_M$ ($m \in \mathbb{Z}_M$ ($\mathbb{Z}$) if $\sigma$ (non)comp.)
- (Generalised) wavelet $\psi$:
  fixed finite set of fns. $\psi \in V$ s.t. its rescaled translates $\psi_m^M$ form an ONB of $W_M = V_M^\perp$ with $V_{\kappa(M)} = V_M \oplus V_M^\perp$ and given $\kappa(M) > M$

- MRA:
  1. Nested system $V_M \subset V_{M'} \subset V = L_2(\sigma),\ M \leq M'$ of "1-particle" HS (smearing fuctions, form factors,...),
  2. "mother scaling function" $\chi$: rescalings/translates $\chi_m^M$ provide ONB of $V_M$.
- Renormalisation wrt MRA [Federbush et. al., TT]:
  Let $L_M \subset \ell_2$ with isometric (bi)injection and projection

$$I_M :\ L_M \to V_M \subset V;\ f_M \mapsto \sum_m f_M(m)\ \chi_m^M \ \Rightarrow\ p_M = I_M\ I_M^\dagger : V \mapsto V_M$$

- Coarse grainig map $I_{MM'} = I_{M'}^\dagger\ I_M$ automatically satisfies consistency due to MRA structure

$$I_{M'}\, I_{MM'} = I_M,\quad I_{M_2 M_3}\ I_{M_1 M_2} = I_{M_1 M_3}$$

- Additional desired features of $\chi$: position and momentum locality, smoothness
  [Cohen, Daubechies, Haar, Meyer, Shannon, ...]

# Hamiltonian renormalisation of Hamiltonian system

- Step 1: Pick MRA structure = coarse graining tool $I_M$
- Step 2: Discretisation of phase space (UV cut-off):

$$\Phi_M := I_M^\dagger \Phi, \quad \Pi_M := I_M^\dagger \Pi$$

- Step 3: Initial Hamiltonian on discretised phase sp. (for differentiable MRA)

$$H_M^{(0)}[\Phi_M, \Pi_M] := H[I_M \Phi_M, \; I_M \Pi_M]$$

- Step 4: Pick initial $(\mathcal{H}_M^{(0)}, \Omega_M^{(0)})$: For $\sigma$ compact, (IR cut-off) $\mathcal{H}_M^{(0)}$ typically unique (Stone – v. Neumann), $H_M^{(0)} \; \Omega_M^{(0)} := 0$ (vacuum), Weyl elements: $w_M[f_M] := \exp(i < f_M, \Phi_M >_{L_M})$, span of $w_M[f_M]\Omega_M^{(0)}$ dense
- Step 5: Renormalisation flow: Iteratively construct $(\mathcal{H}_M^{(n)}, \; \Omega_M^{(n)}, \; H_M^{(n)})$; $M \in \mathcal{M}, \; n \in \mathbb{N}$ s.t. for given $\kappa : \; \mathcal{M} \to \mathcal{M}; \; M' = \kappa(M) > M$ get isometries, Hamiltonian

$$J_{MM'} \; w_M[f_M] \; \Omega_M^{(n+1)} := w_{M'}[I_{MM'} \; f_M] \; \Omega_{M'}^{(n)},; \; H_M^{(n+1)} := J_{MM'}^\dagger \; H_{M'}^{(n)} \; J_{MM'}$$

- Step 6: Fixed points = continuum theory candidates
- Note: Weyl states exited everywhere $\Rightarrow$ q'um non-degeneracy

# Hamiltonian renormalisation of constrained systems

- Idea: Simply copy ren. programme for each constraint $D[u]$ "as if it were a Hamitonian"
- Questions:
  - common vacuum ∃? Necessary?
  - should one also discretise (lapse, shift) test fn $u$, how?
  - how does constraint algebra/anomalies react to renormalisation flow?
- Study those questions for solvable PFT [Kuchar], [Zwicknagel, TT]

# Hamiltonian renormalisation of PFT on the cylinder

- $\sigma = [0,1) = S^1$ compact, periodic bdry cond.
- Lesson 1: Some degree of smoothness of scaling fn. $\chi$ mandatory
- Lesson 2: rapid decrease of Fourier trafo $\hat{\chi}$ mandatory
- Violated for Haar MRA (classic block spin coarse graining)
- E.g. Dirichlet MRA works: $\mathcal{M}$: odd integers, $M \leq M' \quad \Leftrightarrow \quad \frac{M'}{M} \in \mathbb{N}$, $\kappa(M) := 3M$

$$\chi^M_m(x) = \frac{\sin(\pi\; M\; [x - x^M_m])}{\sin(\pi\; [x - x^M_m])},\; x^M_m = \frac{m}{M}; m \in \{0, 1, .., M-1\}$$

- Lesson 3: Ren. flow indeed has known cont. theory as fixed pt.
- Lesson 4: Common vacuum unimportant, $\nexists$ in PFT (Virasoro central extension)
- Lesson 5: natural test fn. discretis. $u_M := p_M u$ possible if MRA diff. but not nec.
- Lesson 6: Finite resolution continuum constraints ("blocked from continuum")
  - must never close
  - physically correct: finite resolution "artefacts" $A_M$: Let $P_M := J_M\; J_M^\dagger$

    $$D_M[u] := P_M\; D[u]\; P_M,\quad [D_M[u], D_M[v]] = -i\; D_M[[u,v]] + \zeta(u,v)\; P_M + A_M(u,v)$$
  - Finite res. anomaly freeness check: $\text{w-lim}_{M\to\infty}\; A_M = 0$

# Take home lessons

- Anomaly freeness of $\mathfrak{h}$ and q'um non-degeneracy are strongly correlated
- natural density weight one: not necessarily obstacle to algebra closure in q'um non-deg. representations
- closure directly on $\mathcal{H}$ not excluded (no habitats)
- eponentiated Ham. constr.: presumably very different action from what was "guessed" so far
- Beautifully demonstrated in Smolin's $U(1)^3$ QG model
- (Hamiltonian) renormalisation:
  - systematises search for q'um non-deg reps.
  - disentangles mere discretisation artefacts from true anomalies