Spacetime Covariance and Propagation in canonical LQG

Madhavan Varadarajan

(Raman Research Institute)

ILQGS March 31, 2020

Rough Plan:

1. **The quantum Hamiltonian constraint**
- Infinitely many choices in its construction
-Constrain choices by non-trivial physical requirements: Sptime cov and Propagation

2. **Spacetime covariance**
- General Remarks
- $U(1)^3$ model

3. **Propagation**
- General Remarks
- $U(1)^3$ model

General Viewpoint

- LQG =Non-Perturbative, Generally covariant quantization of GR. Kinematics well understood. Dynamics in progress.
- Key properties of LQG:
- \- Representation of Triad oprtrs yields a picture of discrete spatial geometry. Area quantized with a smallest non-zero eigen value $\sim$ Planck area.
- \- Due to spatial diffeomorphism invariance, local connection opertor does not exist. Only exponential functions of connections exist as operators. Deep UV degrees of freedom are *not* local connection fields but discrete non-local graphical excitations!
- \- Quantum dynamics is the dyamics of these degrees of freedom.
- Viewpoint is that fundamental theory is somehow discrete and we are probing it thru continuum tools. At some stage will have to jump and confront discreteness on its own terms. But still lot to learn from continuum structures before making educated jump.

The Problem: A brief review

- Quantum dynamics of LQG is driven by Hamiltonian constraint oprtr. Following problem arises in its construction.
- The classical Ham constraint depends on local fields like curvature $F(x)$ of connection. But basic connection operators nonlocal holonomies.
- Classically: F can be extracted from holonomy of a δ size loop through $\lim_{\delta\to 0}\frac{h_{\text{small loop}}-\mathbf{1}}{\delta^2}$.
  QMly: Limit does not exist on operators because background indep Hilbert space cant distinguish between 'smaller' and 'still smaller' loops. Hence proceed as follows.
- Replace all local connection dep fields in Ham constraint by holonomies of small loops of coordinate size δ to get $H_\delta(N)$ which agrees with $H(N)$ in $\delta \to 0$ limit.
- In $H_\delta(N)$ replace holonomies, triads by corresponding operators, get $\hat{H}_\delta(N)$. Take $\delta \to 0$ limit. Hope is that even tho limits of individual bits and pieces dont exist, limit of conglomeration of approximants which make up $\hat{H}_\delta(N)$, exists.

- Can count overall factors of δ in $\hat{H}_\delta(N)$:
  $d^3x \sim \delta^3 \quad \hat{E}^a_i \sim \frac{\widehat{\text{flux}}}{\delta^2} \quad \widehat{\sqrt{q}} \sim \frac{\widehat{\text{Volume}}}{\delta^3} \quad \hat{F}^i_{ab} \sim \frac{\widehat{\text{holonomy}-\mathbf{1}}}{\delta^2}$.
  Density wt 1: no overall factor of δ, can show limit of $\hat{H}_\delta(N)$ exists. But opertr action depends on (infinite) choice of holonomy approximants.
- **Problem**: Action of Ham constraint is infinitely ambiguous.
- **Idea**: Constrain choices through non-trivial physical requirements of spacetime covariance and propagation.
- Quantum Spacetime Covariance:
  Classical Sptime Covariance encoded in characteristic form of constraint algebra (H-K-T). Most non-trivial PB:
  $\{H(M_1), H(M_2)\} = D(\vec{N}_{M_1,M_2,q_{ab}})$. "LHS=RHS"
  In order to implement this sptime covariance condition in quantum theory, look for a choice of Ham constraint operator which yields anomaly free commutators:
  $[\hat{H}(M_1), \hat{H}(M_2)] = i\hbar\hat{D}(\vec{N}_{M_1,M_2,\hat{q}_{ab}})$.

- With density wt 1 constraints:
  RHS $= \int d^3x(M_1\partial_a M_2 - M_2\partial_a M_1)(\frac{E^{ai}E^b_i}{q})E^c_i F_{cbi}$
  $RHS_\delta \sim \quad \delta^3 \quad \times \qquad\qquad\qquad ((\frac{flux}{\delta^2})^2\, (\frac{\delta^3}{Vol})^2)\, (\frac{flux}{\delta^2}\, \frac{hol-1}{\delta^2})$
  $\widehat{RHS}_\delta \sim \delta \times \widehat{finite} \xrightarrow{\delta\to 0} 0$
- $\widehat{LHS}$ must vanish, can vanish for many different actions of $\hat{H}(N)$. How to discriminate?
  Note:Can vanish due to 'wrong reasons'.E.g:If 2nd Ham constraint doesnt act on spin net deformations created by first, M_1, M_2 evaluated on same spin net vertex,commutator vanishes by antisymmetry. I think this hides anomaly: '$M_1\partial M_2$' must come from '$M_1(v)M_2(v+\delta)$'
- In any case, how to use constr algebra to discriminate choices?
  Idea: Use higher density Ham constraints by scaling with powers of $\sqrt{q}$. Then q^{ab} in RHS is scaled by powers of $\sqrt{q} \sim V/\delta^3$, non-trivial $\widehat{RHS}$, non-trivial implementation of sptime covariance.But then singular Ham constr! What to do?

Digression: The Diffeomorphism Constraint Operator:

- In LQG only finite diffeo operators defined on $\mathcal{H}_{kin}$. Can we construct the generator (i.e. diffeo constraint operator) itself by following methods developed for Ham constr?
- Count Powers of δ for $D(\vec{N}) = \int d^3x N^a E^b_i F^i_{ab}$:
  $D_\delta \sim \delta^3 (flux/\delta^2)(hol - 1)/\delta^2 \sim \frac{\underline{Finite}}{\delta}$. 'Singular'
  Despite this one can construct a satisfactory continuum limit.
- One constructs approximant D_δ made up of hol-fluxes so that
  $\hat{D}_\delta(\vec{N})|s\rangle = -i\frac{\hat{U}_{\phi(\vec{N},\delta)} - \mathbf{1}}{\delta}|s\rangle$.
  No cont limit on $\mathcal{H}_{kin}$ but admits cont limit on L-M habitat:
  $\Psi_{[s],f} = \sum_{s\in[s]} f(v_1, .., v_n) < s|$
  $[s]$ =diffeo equivalence class of spin nets with n vertices,
  $f = f(x_1, ..., x_n)$ "vertex smooth function".
  $\Psi_{[s],f}(\hat{D}_\delta(\vec{N})|s\rangle) = \Psi_{[s],g_\delta}(|s\rangle)$.
  $g_\delta \sim \frac{f(\phi^{-1}(\vec{N},\delta)x_1,.,\phi^{-1}(\vec{N},\delta)x_n) - f(x_1,.,x_n)}{\delta}$
- Continuum limit gives state labelled by new function
  $g \sim \sum_{i=1}^n N^a(x_i)\frac{\partial f}{\partial x^a_i}$

Comments:

- Can show that continuum opertr action provides representation of constraint algebra as lie algebra of diffeo grp. Non- triviality of repn rests on kinematically singular constraint oprtr which has limit on a different space (LM habitat).
- To get '(diffeo- 1)/δ' action of $\hat{D}_{\delta}(\vec{N})$ we need to choose holonomy-flux approximants which are attuned to structure of $|s\rangle$. This should have been expected because to move, say, a spin j edge along orbits of shift, we need to somehow 'cancel' its holonomy in repn j and provide a displaced holonomy (also in repn j). So approximants must be tailored to spin and graph labels.
- Could never have guessed correct approximants to be used if we didnt understand what classical constraint generates. (for e.g. curvature approximants depend on fluxes as well!). So need to understand Hamiltonian vector field of classical function well before going to quantization.

  Will use these lessons for Ham constr.

Strategy for Ham constr, commutator in LQG: LHS

- Scale density 1 Ham constr by $\sqrt{q}^{1/3} \sim V^{1/3}/\delta$.
- Look for approximants s.t. $\hat{H}_\delta(N)|s> \sim$
  $\sum_v N(v) \sum_{\mathrm{deformations}(v,\delta)} a_{\mathrm{deform}(v,\delta)} \left(\frac{\hat{U}_{\mathrm{deform}(v,\delta)} - \mathbf{1}}{\delta}\right)|s>$
- Define $\hat{H}_\delta(N)$ on a suitable space of *off shell* states:
  State Ψ_f labelled by function f on Σ.
  $\Psi_f = \sum_{\bar{s}} c_{\bar{s},f} < \bar{s}|$.
  spin net coefficients $c_{\bar{s},f}$ depend on evaluations of f on vertices of $\bar{s}$.
- Evaluate $\Psi_f(\hat{H}_\delta(N)|s>)$. Schematically, want:

- $N\frac{\hat{U}-\mathbf{1}}{\delta}$ to yield contribution
  $N(v)(f(v+\delta) - f(v)/\delta) \xrightarrow[\delta\to 0]{} N\partial f$.
- $\Psi_f(\hat{H}_{\delta'}(M)\hat{H}_\delta(N)|\bar{s}>)|_{\delta',\delta\to 0} \sim M\partial(N\partial f)$.
- LHS: $\Psi_f([\hat{H}(M), \hat{H}(N)]|\bar{s}) \sim (M\partial N - N\partial M)f$.

RHS

- RHS approximant is a single composite operator $\widehat{(RHS)}_{\delta\to 0}$. If classical RHS also was a PB, the approximant would also have δ, δ'. Would be simpler to compare LHS, RHS operators and impose LHS$=$ RHS
- Can do this via miraculous classical identity (CT,MV, 2012).
  Define 'Electric shifts': $N_i^a = \frac{NE_i^a}{q^{\frac{1}{3}}}$.
  $\{H(M), H(N)\} \;=\; \mp 3 \sum_{i=1}^{3} \{D(\vec{M}_i), D(\vec{N}_i)\}$.
  $\mp$: Euclidean/Lorentzian.
  Intriguing fact: Identity not available for density wt 1.
- Status: Replace $SU(2)$ of Euclidean gravity by $U(1)^3$. Model has constr algebra isomrphic to that of Eucl grav.
  (model is same as Smolin's novel $G \to 0$ limit of Eucl grav).
  Can implement above strategy and construct $\hat{H}(N), \widehat{D(\vec{N}_i)}$ such that on suitable state space:
  $[\hat{H}(M), \hat{H}(N)] \;=\; -3 \sum_{i=1}^{3} [\widehat{D(\vec{M}_i)}, \widehat{D(\vec{N}_i)}] \;\neq 0$
  (MV, PRD97, 106007-1 - 106007-89,(2018))

Comments

- Strategy works in $U(1)^3$ model because analysis of class Ham vector fields shows that evolution of connection can be written in terms of combinations of electric field dep diffeos and gauge transformations. Since we have good understanding of phase space indep diffeos and gge transf one can use this to understanding to write down operator correspondents of phase space dep ones as well.
- Detailed calculations on a small space of 'single vertex' distributions. Should be straighfwd to extend to multivertex case as well. There are a few other technical issues. But the main problem is that action of constraint does not support propagation.
- **Status**: With a slight modification of choice of constraint action, modulo some technicalities, one does get vigorous propagation (MV 2019). Anomaly freedom remains to be shown but wrk in progress is promising.

Prognosis

- Next step:Euclidean gravity. Recent analysis of classical eqns show relevant eqns can again be written in terms of such diffeos, gge transf. Roughly, all graph related problems arise in $U(1)^3$ model, here need to confront non-abelian nature of $SU(2)$. Two main open issues:
  (1) Existence of $U(1)^3$ anom-free action consistent with propagation,
  (2) $SU(2)$ generalisatn.
  There is progress in (1) and I feel (2) may be achievable.
- Lorentzian case open. Directly map Eucl solns to Lorentzian ones via TT complexifier?

The $U(1)^3$ model: Some details

- Phase space variables: triplet of $U(1)$ conn-elec fields $A_a^i, E_i^a, i = 1, 2, 3.$
- Constraints:
  $G[\Lambda] = \int \mathrm{d}^3x\ \Lambda^i \partial_a E_i^a$

  $D[\vec{N}] = \int \mathrm{d}^3x\ N^a \left(E_i^b F_{ab}^i - A_a^i \partial_b E_i^b\right)$

  $H[N] = \frac{1}{2} \int \mathrm{d}^3x\ N q^{-1/3} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k$

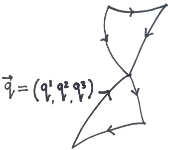
  $F_{ab}^i := \partial_a A_b^i - \partial_b A_a^i \quad , \ q q^{ab} := \sum_i E_i^a E_i^b \quad , \ q = \det q_{ab}.$

  Lapse N density wt $-\frac{1}{3}$.
- Electric Shifts: $N_i^a = N E_i^a q^{-1/3}$
  $D[\vec{N}_i] = \int \mathrm{d}^3x\ N_i^a E_j^b F_{ab}^j$
- Constr algebra (modulo G. Law) isomorphic to that of Eucl GR. On G Law Surface, we have:

  $\{H[N], H[M]\} = (-3) \sum_{i=1}^3 \{D[\vec{N}_i], D[\vec{M}_i]\}$

Quantum Kinematics and Quantum Shift:

- Charge network states:
  $|\alpha, \{\vec{q}\}\rangle \equiv \psi_{\alpha,\{\vec{q}\}}(A) =$ product of edge holonomies.



  Eigen states of electric flux operator. Gge inv $\Rightarrow$ sum of (outgoing) charges at every vertex vanishes

- The Quantum Shift: $\hat{N}^a_i(x) = N\hat{E}^a_i \widehat{\frac{1}{q^{1/3}}}$
  Since $|\alpha, \{\vec{q}\}\rangle$ are eigen states of elec flux, and $\hat{N}^a_i(x)$ only depends on E^a_i, $|\alpha, \{\vec{q}\}\rangle$ are also eigen states of $\hat{N}^a_i(x)$.
  Since $\widehat{\frac{1}{q^{1/3}}}$ non zero only at vertices, quantum shift e.value $\neq 0$ only at chrge net vertices.

The Hamiltonian Constraint: Schematics

- $H(N) = \int \epsilon^{ijk} (\frac{NE^a_i}{q^{\frac{1}{3}}}) F_{abk} E^b_j$
- Action thru diffeos: $N^a F^i_{ab} = \mathcal{L}_{\vec{N}} A^i_b - \partial_b (N^c A^i_c)$.

  'Electric Shift' $N^a_i := \frac{NE^a_i}{q^{\frac{1}{3}}}$.

  $$H(N) = \int \mathcal{L}_{\vec{N}_i} A_{bk}\, \epsilon^{ijk}\, E^a_j + \text{Gauss Law}.$$

  $$\hat{H}(N)\psi(A) = -i \int \mathcal{L}_{\vec{N}_i} A_{bk}\, \epsilon^{ijk}\, \frac{\delta\psi(A)}{\delta A^a_j}$$

  $$\hat{H}_\delta(N)\psi(A) \sim \frac{1}{\delta} \left(\psi(A^j_a + \delta \mathcal{L}_{\vec{N}_i} A_{bk}\, \epsilon^{ijk}) - \psi(A^j_a) \right)$$

  This gives $\hat{U} - \mathbf{1}$ structure.

  Note that "$\hat{U}$" is not pure diffeo since 'internal' indices are being shuffled.
  $\vec{N}_i$ is quantum shift e.value, non-zero only at vertices, 'diffeo' is 'singular'.

  $\hat{H}_\delta(N)$ deforms state at each of its vertices by combination of "singular diffeos + charge flips".

Quantum shift and coordinate dependence

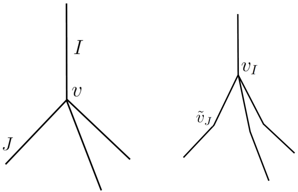
- The Quantum Shift: $N\hat{E}^a_i \widehat{\frac{1}{q^{1/3}}}|\alpha, \{\vec{q}\}\rangle$:
  $\widehat{\frac{1}{q^{1/3}}}$ non zero only at vertices.
  $\hat{E}^a_i(v)$ contributes factors proportional to edge tangents at v.
  $N(v)\hat{E}^a_i(v)\widehat{\frac{1}{q^{1/3}}}(v)|\alpha, \{\vec{q}\}\rangle = N(v)\big(\sum_{I_v} \; q^i_{I_v}\hat{e}^a_{I_v}\big) \;\lambda_v|\alpha, \{\vec{q}\}\rangle$
  So Ham constr acts at vertices and generates graph deformations along each edge tangent. It also flips various charges on the deformed graph.
- Coordinate Dependence: Need coordinate patch to regulate, define quantum shift. $N(v)\hat{e}^a_{I_v}$ **coordinate dependent**. Need to choose coordinate patches at each vertex of each chrge net. Choices should be consistent with obtaining correct commutators with diffeo constraint i.e. choices must be diffeo covariant. Can be done by choosing patches for chrge net and its diffeo image to be related by diffeos. Diff cov is then achieved by tailoring the action of the constraint to properties of the off shell state being acted upon.

Linear Vertices

- Recall structure of constraint: $\hat{H}_\delta(N)|s> \sim$
  $\sum_v N(v) \sum_{\text{deformations}(v,\delta)} a_{\text{deform}(v,\delta)} \left(\frac{\hat{U}_{\text{deform}(v,\delta)} - \mathbf{1}}{\delta}\right)|s>$
- Recall that off shell states are linear superpostns of chrge net bras with coefficients $\sim$ evaluation of function f at vertices. Action of $\hat{H}(N)$ yields new function $\sim N\partial f$ obtained by comparing f at vertex of 'child' $\hat{U}_{deform}|s\rangle$ and at vertex of 'parent' $|s\rangle$.
- For commutator, want action of $\hat{H}(M)$ on the new off shell state. $\hat{H}(M)$ action again in terms of $\hat{U}_{deform} - 1$. Evaluation on new off shell state now involves comparision of $N\partial f$ at vertex of 'child' $\hat{U}_{deform}|s\rangle$ and at vertex of 'parent' $|s\rangle$.
- Different coordinate patches at child, parent vertex. Jacobians on LHS! Similarly also for RHS. Calculation works if Jacobians can pass thru ∂. Want *constant* Jacobians, linear coord transf. Achieved if all bra summands of off shell states have Linear Vertices.

Linear Vertices and Conical Deformations

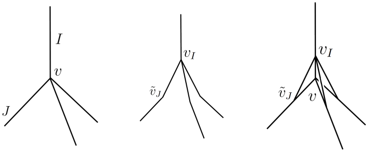
- Vertex linear if there is coordinate patch in which all edges appear as straight lines ("higher moduli vanish"). Call such coordinates 'linear coordinates'. Use them to evaluate q.shift.
- Wrto these coord visualise deformation generated by quantum shift as abrupt pulling of vertex structure along I_vth edge within δ size nbrhood of v. Result is 'conical deformation' wherein remaining $N-1$ edges pulled abruptly in dirn of the I_vth one,



- Displaced vertex has same valence N as parental one: '$N \rightarrow N$' deformation. Note that deformed vertex also linear.
- Calculation hints at PL discrete structure. Similar to simplest Spin Foam picture?

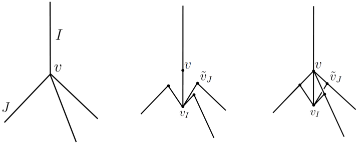
Deformed Hamiltonian Children

Downward deformations:



undef graph def graph def child

Upward deformations:



undef graph def graph def child

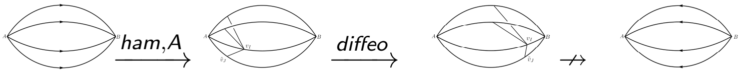
NOTE: Deformed Electric diffeo children live on the deformed graph with original charges.

NOTE: 2nd action on deformation generated by first!

Propagation:

- Recall: Hamiltonian constructions in LQG lead to constraint actions at vertices of spin nets. These actions are obtained through a $\delta \to 0$ 'continuum limit' of actions of Hamiltonian constraint approximants $\hat{H}_\delta(N)$.
- $\hat{H}_\delta(N)$ is ordered so that it acts only at vertices of spin net state. Action deforms the structure in a δ size neighbourhood of the vertex. Action at one vertex is completely independent of action at another: "Ultralocality".
- In a very influential and beautiful paper in the 90s Smolin provided the first clear articulation of propagation in LQG and its potential tension with ultralocality. He also argued that canonical LQG methods were not consistent with propagation of quantum perturbations from one vertex of a spin net to another.

- However his detailed arguments are based on Thiemann's claims in the QSD papers which were only available in preprint form. While the history is tangled, the main point is that the conclusion and ensuing folklore that ultralocality of constraint action precludes propagation is incorrect (MV 2017, 2019 in toy models, TT-MV in LQG context, in preparation).
- Folkore seems to be based on fact that while action of $\hat{H}_\delta(N)$ can create/split a vertex for all small enough δ it cannot *merge* 2 vertices in the $\delta \rightarrow 0$ limit.



s $\qquad$ $s_{pert,A}$ $\qquad$ s'

The notion of propagation

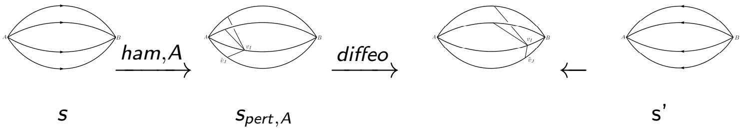
- The intuitive notion of propagation above can be made precise as follows. Any physical state Ψ lies in the kernel of the constraints. Viewing at as an element of the (algebraic) dual, it admits an expansion in terms of spin net states: $\Psi = \sum_s a_s < s|$ Call the set of (ket correspondents of) spin net bras (for which $a_s \neq 0$)as the Ket Set S_{ket}.
  We shall say that Ψ encodes propagation if elements of the Ket Set are related by propagation i.e. there is a subset of elements which form a sequence
  " $|s> \xrightarrow{\text{pert at vertex } v_1} |s_{v_1}> \xrightarrow{\text{prop of pert to vicinity of nbr } v_2} |s'_{v_1}> \xrightarrow{\text{absorptn of pert by } v_2} |s_{v_1 v_2}> \xrightarrow{\text{prop of pert past } v_2} |s'_{v_1 v_2}>$...."
  The longer the sequence is and the more the number of such sequences, the more vigorous is the propagation.

- This notion of propagation in terms of physical states is logically distinct from that deriving from repeated action of constraints on a spin net state. It is conceivable that even if the latter does not generate propagation due to ultralocality, propagation is still encoded in physical states. We shall use the physical state based notion.Of course the detailed structure of the constraint operators determines their kernel so that the encoding of propagation *is* tied to the form of the constraint operators. It is in this sense that demanding propagation constrains the available choices of Ham constr operator.
- One possible route to propagation is as follows. Let the action of $\hat{H}_{\delta}(N)$ on a 'parent' spin net create a set of deformed 'children' spin nets. Let the structure of the Ket Set associated with a physical state be such that:
  1. If a parent is in S_{ket} so are all its children.
  2. If a child is in S_{ket} so are all its possible parents.

  Then propagation can ensue as follows thru possible parents.

Propagation thru Possible Parents:

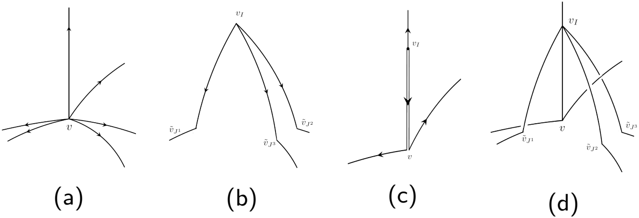
Smolin's conclusions pertain to deformations generated by the constraints i.e. to all children of a given state $|s>$. These do not encode propagation sequences because of the impossibility of the vertex merging operation. However including possible parents *can* and does allow for propagation sequences!



Note that we have used the fact that physical states are diff inv to move perturbation from vicinity of 1 vertex to next. This illustrates the haziness of the ultralocality concept in a diff inv setting. In PFT it is $\hat{U} - 1$ constr structure which implements properties 1., 2. In $U(1)^3$ modulo some technicalities, also this is true.
NOTE: To bypass Lee's arguments, the existence of possible parents is crucial: even if they dont manifest as summands, it is crucial that their children do. Non-unique parentage is necessary.

Propagation in $U(1)^3$:$N \to 4$ deformations

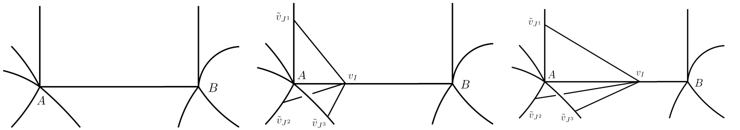
- Consider graph with neighboring vertices v_1, v_2 of valence N_1, N_2. $N \to N$ defrmtn at vertex v_1 gives child vertex of valence N_1. If $N_2 \neq N_1$, deform of v_2 could nevr give such child vertex. So $N \to N$ deformation cannot propagate deformations between vertices of distinct valence.
- Hence define a slight $N \to 4$ modification of the constr action wherein at a time only 3 of the edges at each vertex were deformed along a 4th:



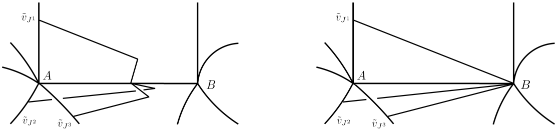
- Can build constraint action based on such deformations. These are consistent with propagation and work in progress with regard to anomaly freedom seems promising.

Propagation sequence in $U(1)^3$:

We need to show that there exist propagation sequences relating elements of the Ket Set. We illustrate one such sequence below for the smallest Ket Set which is subject to the properties discussed earlier and which contains the parental ket (a):



(a) (b) (c)



(d) (e)

Propagation in $U(1)^3$:

A 'foamy' view

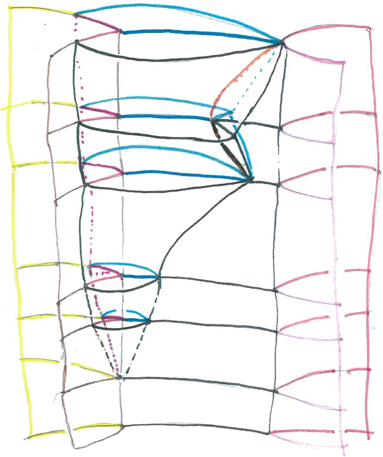


Figure attempts to show a rough qualitative 2+1 analog. New vertices are depicted as 3 valent not 4 valent.

Summary

- Over the last decade or so have tried to use the requirements of anom free constr algebra and of propagation to home in onto the physically correct Ham constr operator.
- Progress, in toy models of increasing complexity (PFT, HK model≡Diffeo constraint, $U(1)^3$ model), is due to the fact that classical evolution can be understood in terms of spatial diffeos. In $U(1)^3$ diffeos are electric field dependent.
- We are hopeful of progress in Euclidean theory because evolution eqns (Abhay 2019 (with a little help from MV)) can be written as:
  $\dot{E}^a{}_i = -\epsilon_i{}^{jk}\mathbb{L}_{\vec{N}_j} E^a_k$
  $\dot{B}^a{}_i = -\epsilon_i{}^{jk}\mathbb{L}_{\vec{N}_j} B^a_k,\ B^a_i = \eta^{abc}F_{bci}$
  $\mathbb{L}_{\vec{N}_j}$ is exactly $\mathcal{L}_{\vec{N}_j}$ with ordinary derivatives replaced by gauge covariant ones.
- $$\begin{aligned}\dot{A}^i_a &= -\epsilon^{ij}{}_k(\mathcal{L}_{\vec{N}_i} A^k_a - \partial_a(N^b_j A^k_b) - \epsilon^k_{pq} A^p_a (N^b_j A^q_b)) \\ &= -\epsilon^{ij}{}_k(\mathcal{L}_{\vec{N}_i} A^k_a - \mathcal{D}_a N^b_{(j)} A^k_b)\end{aligned}$$