The spinfoam action

International LQG Seminar

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16 September 2014

Outline of the talk

I present an action for simplicial gravity in first-order area-connection variables. The theory has a Hamiltonian and local gauge symmetries. Generic solutions represent twisted geometries, and have curvature – there is a deficit angle around triangles.

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Some key references:

*ww, New action for simplicial gravity in four dimensions (2014), arXiv:1407.0025.
*ww, One-dimensional action for simplicial gravity in three dimensions, Phys. Rev. D 90 (2014), arXiv:1402.6708.
*ww, Hamiltonian spinfoam gravity, Class. Quant. Grav. 31 (2014), arXiv:1301.5859.
*M Cortês and L Smolin, Spin foam models as energetic causal sets (2014), arXiv:1407.0032.
*L Freidel and S Speziale, From twistors to twisted geometries, Phys. Rev. D 82 (2010), arXiv:1006.0199.

The spinfoam approach

What are spinfoams?

We have to solve the Wheeler–De Witt constraint:

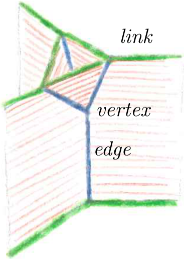
$$\hat{H}_{\mathrm{WdW}}\Psi = 0. \tag{1}$$

Idea: Find the kernel of the WdW equation by a path integral approach:

$$\hat{H}_{\mathrm{WdW}} Z_{\mathcal{M}} = 0: \quad Z_{\mathcal{M}}[h_{ab}] = \int_{g_{ab}|\partial\mathcal{M}=h_{ab}} \mathcal{D}\mu[g], \; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[g]} \tag{2}$$

on a simplicial decomposition $\mathcal{S}$ of $\mathcal{M}$:

$$Z_{\mathcal{S}}[\vec{\jmath},\vec{\imath}] = \sum_{j_f, i_e} \prod_{f:\text{faces}} \prod_{v:\text{vertices}} \mu(j_f) A\big[j_{v_1},\ldots,j_{v_{10}}, i_{e_1},\ldots,i_{e_5}\big]. \tag{3}$$



Where j_f and i_e are spins and intertwiners describing the four-dimensional continuation of LQG boundary states into the bulk.

*JB Hartle and SW Hawking, Wave function of the Universe, Phys. Rev. D 28 (1983).
*MP Reisenberger and C Rovelli, “Sum over Surfaces” form of Loop Quantum Gravity, Phys. Rev. D 56 (1997), `arXiv:gr-qc/9612035`.

Results and Debates

The EPRL model is a concrete proposal realizing this idea.

Interesting results:

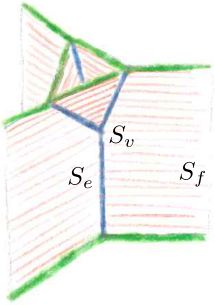
- Graviton propagator, Regge-action for large spins, inclusion of a (positive) cosmological constant, addition of fermions and Yang–Mills fields, spinfoam cosmology, horizon thermodynamics…

But also ongoing debates:

- How can we find continuum GR? Which limit? Summing, refining?
- Do we miss additional secondary (torsional) constraints?
- Is there a flatness problem?
- Is there a notion of causality in spinfoams?

I think, these issues have little to do with the quantum theory itself. We should address these question already at the classical level. What we need is a framework of simplicial gravity in area–connection variables.

*A Perez, The Spin-Foam Approach to Quantum Gravity, Living Rev. Relativity 16 (2013), `lrr-2013-3`.
*J Engle, ER Livine, R Pereira and C Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. B (2008), `arXiv:0711.0146`.



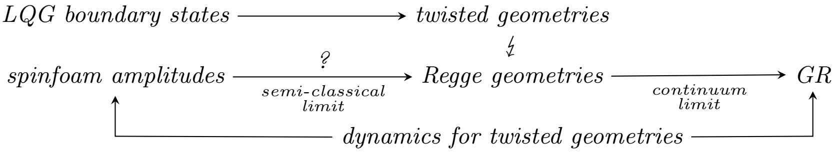
- A *local* spinfoam model assigns amplitudes A_e, A_f, A_v,… to the elementary building blocks of the simplicial complex.
- In the semi-classical limit these amplitudes turn into action functionals: $A_e \propto \mathrm{e}^{\mathrm{i}S_e}$, $A_f \propto \mathrm{e}^{\mathrm{i}S_f}$, $A_v \propto \mathrm{e}^{\mathrm{i}S_v}$,…

Can we write down a spinfoam action $\sum_e S_e + \sum_f S_f + \sum_v S_v + \dots$ with the same field-content as LQG? A theory of simplicial gravity in terms of Ashtekar–Barbero variables?

See also:

*L Freidel and J Hnybida, A discrete and coherent basis of intertwiners, Class. Quantum Grav. 31 (2014), arXiv:1305.3326.
*B Dittrich and P Höhn, Constraint analysis for variational discrete systems, J. Math. Phys. 54 (2013), arXiv:1303.4294.

A change of perspective



Tension between LQG kinematics and dynamics

- The LQG boundary states represent twisted geometries: Every tetrahedron has a unique volume, and every triangle has a unique area, yet there are no unique edge lengths.
- A conceptual tension: We always try to find just Regge gravity in the semi-classical limit. Yet, our kinematical framework is more general: Twisted geometries are less restrictive than Regge geometries.
- Key question: Can we formulate the dynamics of discretized gravity in terms of twisted geometries?

The spinfoam action

Plebański principle

The BF action is topological, and determines the symplectic structure of the theory:

$$S_{\mathrm{BF}}[\Sigma, A] = \frac{\hbar}{2{\ell_{\mathrm{P}}}^2} \int_M \left(* \Sigma_{\alpha\beta} - \beta^{-1} \Sigma_{\alpha\beta} \right) \wedge F^{\alpha\beta}[A] \equiv \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta}. \quad (4)$$

General relativity follows from the simplicity constraints added to the action:

$$\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}. \quad (5)$$

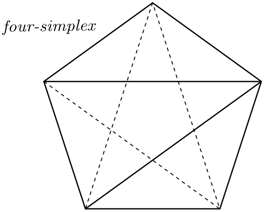
With the solutions:

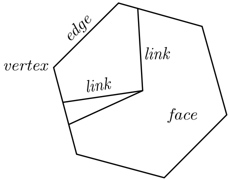
$$\Sigma^{\alpha\beta} = \begin{cases} \pm e_\alpha \wedge e_\beta, \\ \pm * (e_\alpha \wedge e_\beta). \end{cases} \quad (6)$$

Notation:

- $\alpha, \beta, \gamma \dots$ are internal Lorentz indices.
- ${\Sigma^\alpha}_\beta$ is an $\mathfrak{so}(1,3)$-valued two-form.
- ${A^\alpha}_\beta$ is an $SO(1,3)$ connection, with ${F^\alpha}_\beta = \mathrm{d}{A^\alpha}_\beta + {A^\alpha}_\mu \wedge {A^\mu}_\beta$ denoting its curvature.
- e^α is the tetrad, diagonalizing the four-dimensional metric $g = e_\alpha \otimes e^\alpha$.
- ${\ell_{\mathrm{P}}}^2 = 8\pi\hbar/Gc^3$, and β is the Barbero–Immirzi parameter.

Discretized BF theory with spinors on a lattice





We can write the discretized BF action as a sum over the two-dimensional simplicial faces $f_1, f_2, \dots$:

$$\begin{aligned} S_{\mathrm{BF}}[Z_{f_1}, Z_{f_2}, \dots; \utilde{Z}_{f_1}, \utilde{Z}_{f_2}, \dots; \zeta_{f_1}, \zeta_{f_2}, \dots; \Lambda_{e_1}, \Lambda_{e_2}, \dots] &= \sum_{f:\mathrm{faces}} S_f \\ &= \sum_{f:\mathrm{faces}} \oint_{\partial f} \Big[\pi^f_A D\omega^A_f - \utilde{\pi}^f_A \mathrm{d}\utilde{\omega}^A_f + \zeta_f \big(\pi^f_A \omega^A_f - \utilde{\pi}^f_A \utilde{\omega}^A_f \big) \Big] + \mathrm{cc.} \end{aligned} \tag{7}$$

Notation:

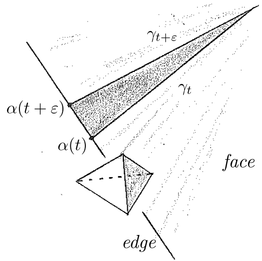
- $A, B, C, \dots$ are spinor indices, and $\mathrm{cc.}$ denotes complex conjugation.
- Each face f carries two twistors: $Z_f, \utilde{Z}_f : \partial f \to \mathbb{T} \simeq \mathbb{C}^4$, $Z = (\bar{\pi}_{A'}, \omega^A)$.
- $\zeta_f : \partial f \to \mathbb{C}$ is a Lagrange multiplier imposing the constraint $\varDelta_f = \utilde{\pi}_A \utilde{\omega}^A - \pi_A \omega^A$.
- D is the covariant differential, $\dot{e}$ an edge's tangent vector: $\dot{e} \lrcorner D\pi^A = \dot{\pi}^A + [\Lambda_e]^A{}_B \pi^B$.

- Step 1: Discretize the action:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \approx \sum_{f:\mathrm{faces}} \int_{\tau_f} \Pi_{\alpha\beta} \int_f F^{\alpha\beta} \equiv \sum_{f:\mathrm{faces}} S_f .$$

- Step 2: Define the smeared flux:

$$\Pi_f^{\alpha\beta}(t) = \int_{\tau_f} \mathrm{d}x\, \mathrm{d}y\, [h_{\gamma(t,x,y)}]^{\alpha}{}_{\mu} [h_{\gamma(t,x,y)}]^{\beta}{}_{\nu} \left[\Pi_{p(x,y)}(\partial_x, \partial_y)\right]^{\mu\nu} .$$



- Step 3: Employ the non-Abelian Stoke's theorem:

$$\int_{\gamma_t} \mathrm{d}z\, h^{-1}_{\gamma_t(z)} F_{\gamma_t(z)}(\partial_z, \partial_t) h_{\gamma_t(z)} = h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)},$$

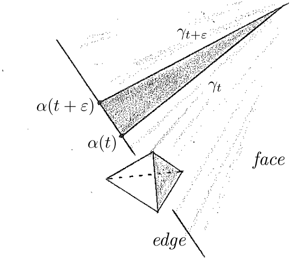
to eventually find the one-dimensional action:

$$S_f = -\int_{\partial f} \mathrm{d}t \left[h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)}\right]_{\alpha\beta} \Pi_f^{\alpha\beta}(t).$$

- Step 4: Introduce spinors to diagonalize both holonomies and fluxes:

$$\Pi_f^{\alpha\beta}(t) = \frac{1}{2}\bar{\epsilon}^{A'B'}\omega_f^{(A}(t)\pi_f^{B)}(t) + \text{cc.},$$

$$[h_{\gamma_t}]^A{}_B = \text{Pexp}\Big(-\int_{\gamma_t} A\Big)^A{}_B = \frac{\underset{\sim}{\omega}_f^A(t)\pi^f_B(t) - \underset{\sim}{\pi}_f^A(t)\omega^f_B(t)}{\sqrt{E_f(t)}\sqrt{\underset{\sim}{E}_f(t)}}.$$



We also need the area-matching constraint:

$$\varDelta_f := \underset{\sim}{\pi}^f_A\underset{\sim}{\omega}_f^A - \pi^f_A\omega_f^A \equiv \underset{\sim}{E}_f(t) - E_f(t).$$

Putting the pieces together yields the face action:

$$S_f[Z, \underset{\sim}{Z}, A, \zeta] = = \int_{\partial f} \mathrm{d}t \Big[\pi_A \frac{D}{\mathrm{d}t}\omega^A - \underset{\sim}{\pi}_A \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{\omega}^A - \zeta\varDelta\Big] + \text{cc.} \qquad (9)$$

Instead of discretizing the quadratic simplicity constraints

$$\Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu}, \tag{10}$$

we will use the linear simplicity constraints:

For a tetrahedron T_e (dual to an edge e) there exist an internal future-oriented four-normal n_e^α such that the fluxes through its four bounding triangles τ_f (dual to a face f: $e \subset \partial f$) annihilate n_e^α:

$$\int_{\tau_f} \Sigma_{\alpha\beta} n_e^\beta = 0. \tag{11}$$

The spinorial parametrization turns the simplicity constraints into the following complex conditions:

$$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0, \tag{12a}$$

$$W_{ef} = n_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \stackrel{!}{=} 0. \tag{12b}$$

Adding the simplicity constraints

- The simplicity constraints reduce the $SO(1,3)$ spin connection $A^\alpha{}_\beta$ to the $SU(2)_n$ Asthekar–Barbero connection:

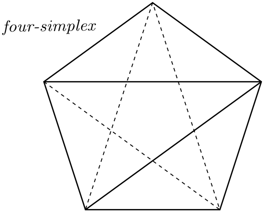
$$\mathcal{A}^\alpha = n^\mu \Big[\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\rho} A^\nu{}_\rho + \beta A^\alpha{}_\mu\Big]. \tag{13}$$

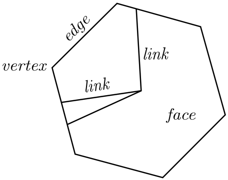
- We introduce Lagrange multipliers $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ and get the following constrained action for each face in the discretization:

$$S_{\text{face}}[Z, \underset{\sim}{Z}|\zeta, z, \lambda|\mathcal{A}, n] = \oint_{\partial f} \Big(\pi_A \mathcal{D}\omega^A - \underset{\sim}{\pi}_A \mathrm{d}\underset{\sim}{\omega}^A - \zeta\big(\underset{\sim}{\pi}_A \underset{\sim}{\omega}^A - \pi_A\omega^A\big) + \\ - \frac{\lambda}{2}\Big(\frac{\mathrm{i}}{\beta + \mathrm{i}}\pi_A\omega^A + \text{cc.}\Big) - z\, n^{AA'}\pi_A\bar{\omega}_{A'}\Big) + \text{cc.}, \tag{14}$$

  where $\mathcal{D}\pi^A = \mathrm{d}\pi^A + \mathcal{A}^\alpha \tau^A{}_{B\alpha}\pi^B$ is the $SU(2)_n$ covariant differential.
- Problem: There is no term in the action that would determine the t-dependence of the normal n^α_e along the edges $e(t)$.
- We now have to make a proposal.

Four-dimensional closure constraint





Any proposal for the dynamics of the time normals must respect the closure constraint at the vertices (four-simplices):

We define the volume-weighted four-normal:

$$p^e_\alpha = n^e_\alpha \mathrm{Vol}(e). \tag{15}$$

At every four simplex we have the closure constraint:

$$\sum_{\substack{\text{outgoing edges } e \\ \text{at } v}} p^e_\alpha = \sum_{\substack{\text{incoming edges } e \\ \text{at } v}} p^e_\alpha . \tag{16}$$

Notation:

- $\mathrm{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \mathrm{cc}.$

Both the Gauß constraint $\sum_\tau \Sigma_{\alpha\beta}[\tau] = 0$ and the four-dimensional closure constraint are a result of the vanishing of torsion:

$$D\wedge e^\alpha = 0, \quad \text{implies also :} \tag{17}$$

$$\frac{1}{3!} D\wedge(\epsilon_{\alpha\beta\mu\nu} e^\beta \wedge e^\mu \wedge e^\nu) = 0. \tag{18}$$

This is a four-form, its integral over a four-simplex yields:

$$\sum_{\substack{\text{outgoing edges } e \\ \text{at } v}} p^e_\alpha = \sum_{\substack{\text{incoming edges } e \\ \text{at } v}} p^e_\alpha. \tag{19}$$

For the tetrahedron T_e dual to the edge e:

$$p^e_\alpha = \pm\frac{1}{3!}\int_{T_e} \epsilon_{\alpha\beta\mu\nu} e^\beta \wedge e^\mu \wedge e^\nu. \tag{20}$$

The proposal for the dynamics of the time-normals

Any proposal for the dynamics of the time-normals
- must respect the four-dimensional closure constraint, and
- be consistent with all symmetries of the action.

The following action fulfills these requirements:

$$S_{\mathrm{edge}}[X, p|N, \mathrm{Vol}(e)] = \int_e \Big(p_\alpha \mathrm{d}X^\alpha - \frac{N}{2}\big(p_\alpha p^\alpha + \mathrm{Vol}^2(e)\big)\Big). \qquad (21)$$

We just need an additional boundary term at the vertices:

$$S_{\mathrm{vertex}}[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}] = \sum_{e:e\ni v} \big(Y_v^\alpha - X_{ev}^\alpha\big) v_\alpha^{ev}. \qquad (22)$$

Where N is a Lagrange multiplier imposing the mass-shell condition:

$$C := \frac{1}{2}\Big(p_\alpha p^\alpha + \mathrm{Vol}^2(e)\Big) \stackrel{!}{=} 0. \qquad (23)$$

Putting the pieces together – defining the action

Adding the face, edge and vertex contributions gives us a proposal for an action for discretized gravity in first-order variables:

$$
\begin{aligned}
S_{\text{spinfoam}} = \sum_{f:\text{faces}} & S_{\text{face}}\big[Z_f, \underset{\sim}{Z}_f \big| \zeta_f, z_f, \lambda_f \big| \mathcal{A}_{\partial f}, n_{\partial f}\big] + \\
+ \sum_{e:\text{edges}} & S_{\text{edge}}\big[X_e, p_e \big| N_e, \mathrm{Vol}(e)\big] + \\
+ \sum_{v:\text{vertices}} & S_{\text{vertex}}\big[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}\big].
\end{aligned}
\tag{24}
$$

Notation:

- Z_f and $\underset{\sim}{Z}_f$ are the twistors $Z_f : \partial f \rightarrow \mathbb{T} \simeq \mathbb{C}^4$ parametrizing the $SL(2,\mathbb{C})$ holonomy-flux variables.
- ζ_f, λ_f and z_f are Lagrange multipliers imposing the area-matching constraint and simplicity constraints respectively.
- $\mathcal{A}$ is the $SU(2)_n$ Ashtekar–Barbero connection along the edges of the discretization.
- n denotes the time normal of the elementary tetrahedra.
- p_e is the volume-weighted time-normal, of the tetrahedron dual to the edge e.
- $\mathrm{Vol}(e)$ denotes the corresponding three-volume.
- N is a Lagrange multiplier imposing the mass-shell condition $C = 0$.

Hamiltonian formulation, twisted geometries and curvature

Three immediate tests for the model

1. Is there a Hamiltonian formulation of the dynamics of the theory?
2. What kind of four-dimensional geometries do the equations of motion generate?
3. Does the model have curvature?

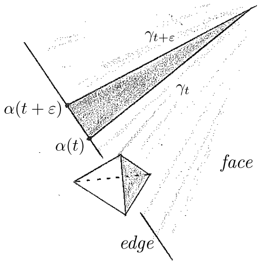
The Hamiltonian:

$$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + z^f W_{ef} + \bar{z}^f \bar{W}_{ef} + \lambda^f V_f \right) + N C_e, \quad (25)$$

generates the t-evolution along the edges of the discretization:

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_f^A = \{H, \omega_f^A\}. \quad (26)$$

The fundamental Poisson brackets are:



$$\{p^e_\alpha, X^\beta_e\} = \delta^\beta_\alpha,$$

$$\{\pi^f_A, \omega^B_{f'}\} = +\delta_{ff'}\delta^B_A, \qquad \{\bar{\pi}^f_{A'}, \bar{\omega}^{B'}_{f'}\} = +\delta_{ff'}\delta^{B'}_{A'},$$

$$\{\underset{\sim}{\pi}^f_A, \underset{\sim}{\omega}^B_{f'}\} = -\delta_{ff'}\delta^B_A, \qquad \{\underset{\sim}{\bar{\pi}}^f_{A'}, \underset{\sim}{\bar{\omega}}^{B'}_{f'}\} = -\delta_{ff'}\delta^{B'}_{A'}.$$

Dirac analysis

- The Hamiltonian preserves all constraints provided $z_f = 0$.
- There are no secondary constraints.

Physical Hamiltonian

$$H_{\mathrm{phys}} = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + \lambda^f V_f \right) + NC. \tag{28}$$

second-class simplicity constraint: $W_{ef} = n_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \stackrel{!}{=} 0,$

first-class simplicity constraint: $V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0,$

area-matching condition (first-class): $\Delta_f = \underset{\sim}{\pi}^f_A \underset{\sim}{\omega}^A_f - \pi^f_A \omega^A_f \stackrel{!}{=} 0,$

mass-shell condition (first-class): $C_e = \frac{1}{2}\left(p^e_\alpha p^\alpha_e + \mathrm{Vol}^2(e) \right) \stackrel{!}{=} 0,$

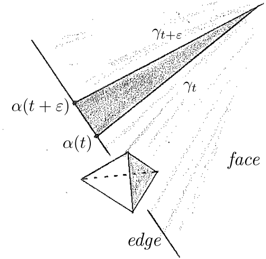
$SU(2)_n$ Gauß constraint (first-class): $G^e_\alpha = \sum_{f:\partial f \supset e} \tau^{AB}{}_\alpha \omega^f_A \pi^f_B + \mathrm{cc.}$

Notation:

- $\tau^A{}_{B\alpha}$ are the $SU(2)_n$ generators: $[\tau_\alpha, \tau_\beta] = n^\mu \epsilon_{\mu\alpha\beta}{}^\nu \tau_\nu$.
- $\mathrm{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \mathrm{cc.}$

Twisted geometries

What kind of four-dimensional geometries does the Hamiltonian generate?



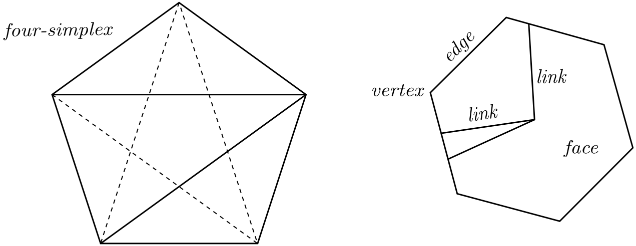
- The simplicity constraints guarantee that the fluxes $\int_{\tau_f} \Sigma_{\alpha\beta}$ define planes in internal Minkowski space.
- The Gauß constraint tells us that these planes close to form a tetrahedron.
- The physical Hamiltonian $H_{\rm phys}$ deforms the shape of the tetrahedron.

The Hamiltonian generates twisted geometries, the relevant term is the mass-shell condition:

$$C = \frac{1}{2}\big(p_\alpha p^\alpha + \mathrm{Vol}^2\big). \qquad (30)$$

$\mathrm{Vol}^2 \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$ preserves the area of the four bounding triangles, and the volume of the tetrahedron, yet it does not preserve the tetrahedron's shape – the Hamiltonian generates a shear.

*E Bianchi, HM Haggard, Bohr-Sommerfeld Quantization of Space, Phys.Rev. D 86 (2012), `arXiv:1208.2228`.



Inter-tetrahedral angles:

$$\cosh \Xi_{vf} = -\eta^{\mu\nu} n^{e}_{\mu} n^{e'}_{\nu}, \quad \text{with: } e \cap e' = v, \text{ and: } e,\, e' \subset \partial f. \tag{31}$$

Deficit angle around a triangle:

$$\Xi_f := \sum_{v\text{: vertices in } f} \Xi_{vf} = \frac{2}{\beta^2 + 1} \oint_{\partial f} \lambda_f. \tag{32}$$

Conclusion

Key features of the spinfoam action

- Branched-continuity: The action is an integral over the entire system of edges, an action for a one-dimensional branched manifold (c.f. relative locality).
- Causal spinfoams: The edges are oriented, we can distinguish future-pointing from past-pointing edges.
- Twisted geometries: The Hamiltonian generates twisted geometries.
- Particle picture: Every tetrahedron represents a massive particle, with internal $SU(2)_n$ little-group color DOFs. The particles' momenta p are the volume-weighted four-normals $p = \mathrm{Vol}\, n$. Mass turns into quantized volume.

Thank you

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