Quasi-local energy from LQG boundary modes

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Basic motivation

… what is a gravitational subsystem?

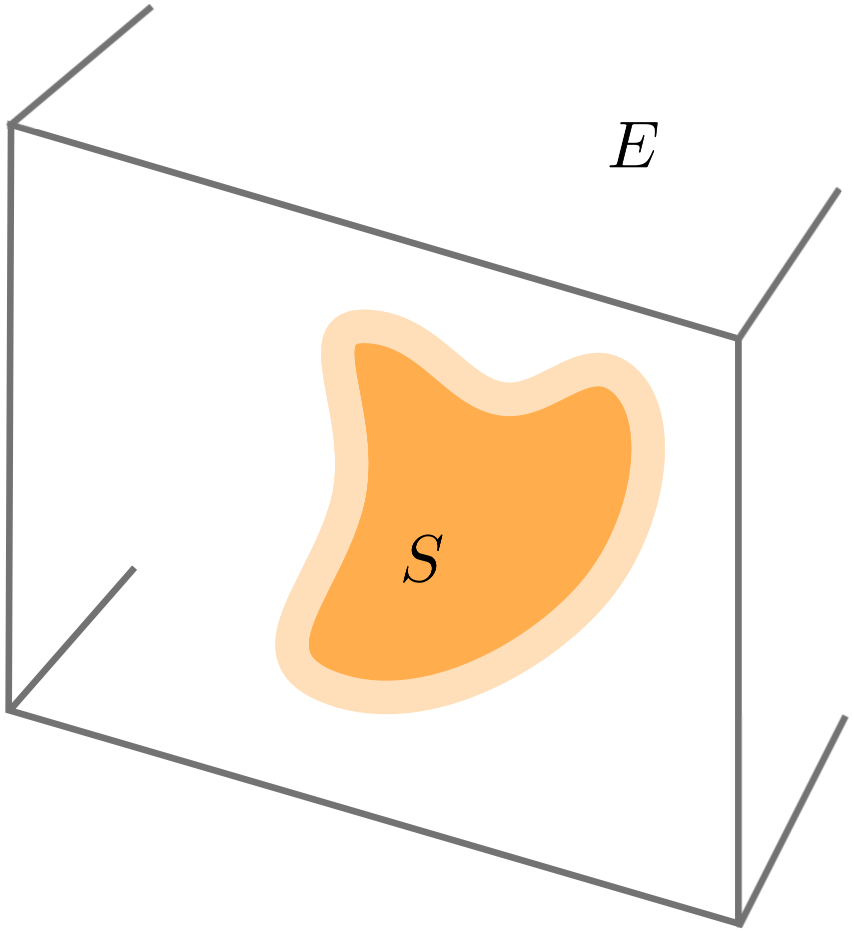
Coarse-graining and GR

- In any realistic experiment, the environment is split into subsystems that evolve for sufficient time in controlled isolation. Such subsystems are characterised by a relatively small number of coarse grained observables.
- One of the main conceptual difficulties in classical and quantum GR is that there is no obvious such coarse graining on the ADM phase space. Some manifestations of this difficulty:
    - *Averaging problem in cosmology* [Buchert, Carfora, Wald, Green, Wiltshire,…].
    - *Problem of time in QG* [Rovelli, Smolin, Kuchař, Thiemann,…].
    - *How to effectively build Dirac observables for GR? What is a thermodynamical ensemble of spacetimes?* [Rovelli, Dittrich, Kuchař, Thiemann,…].
- **Main strategy here:** revers the problem and build the entire phase space from the bottom up.

Subsystem Hamiltonians

- Consider a system coupled to its environment.

$$S = \int \Big[P\mathrm{d}Q - H_E^o[P,Q] + \\ + p\,\mathrm{d}q - H_S^o[p,q] - V[p,q|P,Q] \Big].$$



- The subsystem is characterised by
  - symplectic structure $\Omega_S = \mathrm{d}p \wedge \mathrm{d}q$.
  - Hamiltonian $H_S[p,q|J] = H_S^o[p,q] + V[p,q|J]$.
- Hamilton's equations contain explicit dependence on the environment.

$$\delta[H_S] = \Omega_S(\delta, \partial_t) + \frac{\delta V}{\partial J}\delta J, \qquad J = (P,Q).$$

quasi-local energy

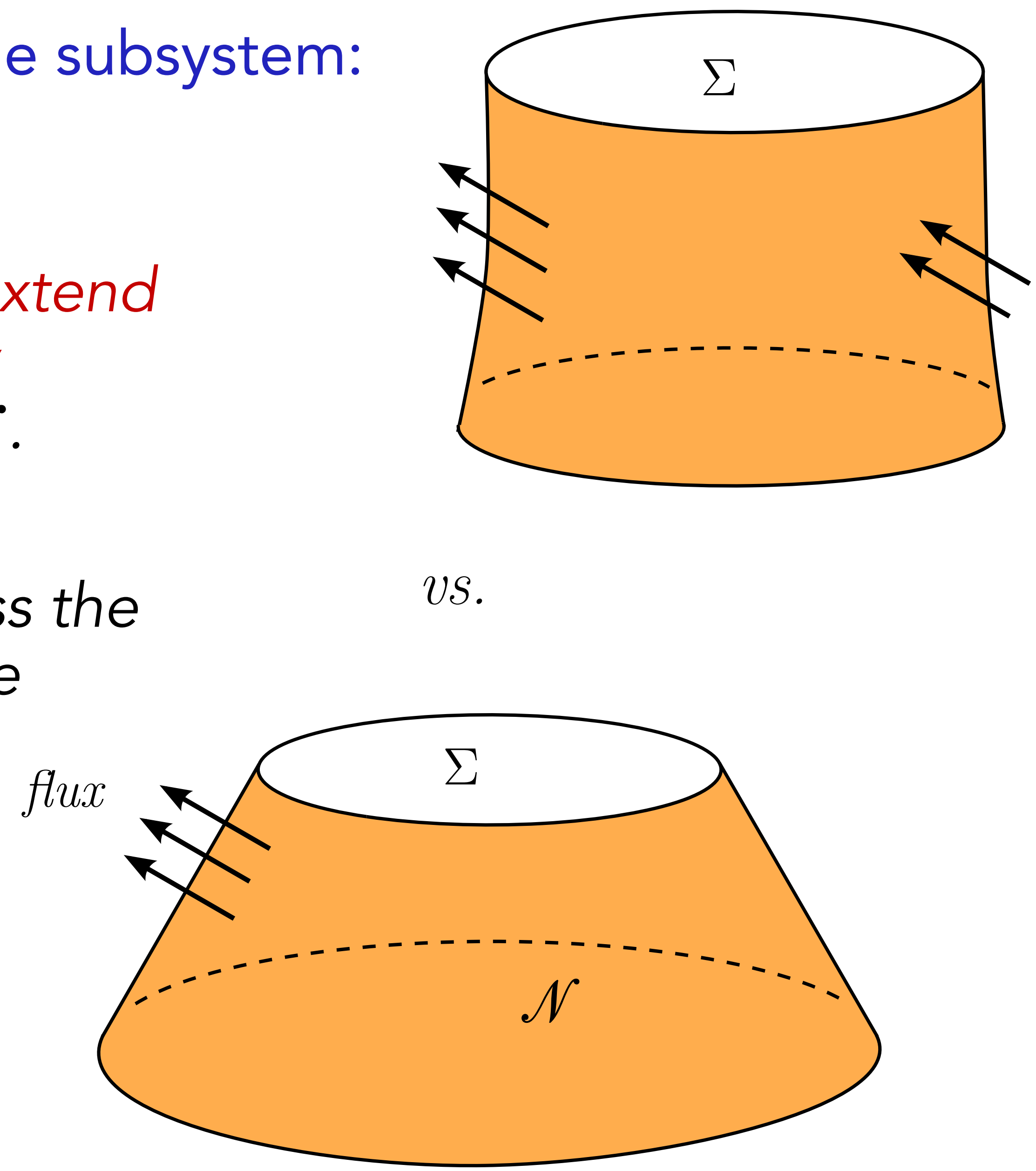
… let us now consider the problem in gravity

Which subsystems?

Two important points that determine the subsystem:

- *Choice must be made for how to extend the boundary of the partial Cauchy hypersurface* $\boldsymbol{\Sigma}$ *into a worldtube* $\mathcal{N}$*.*

- *Flux of gravitational radiation across the worldtube of the boundary must be prescribed as an external source J (background field, c-number).*

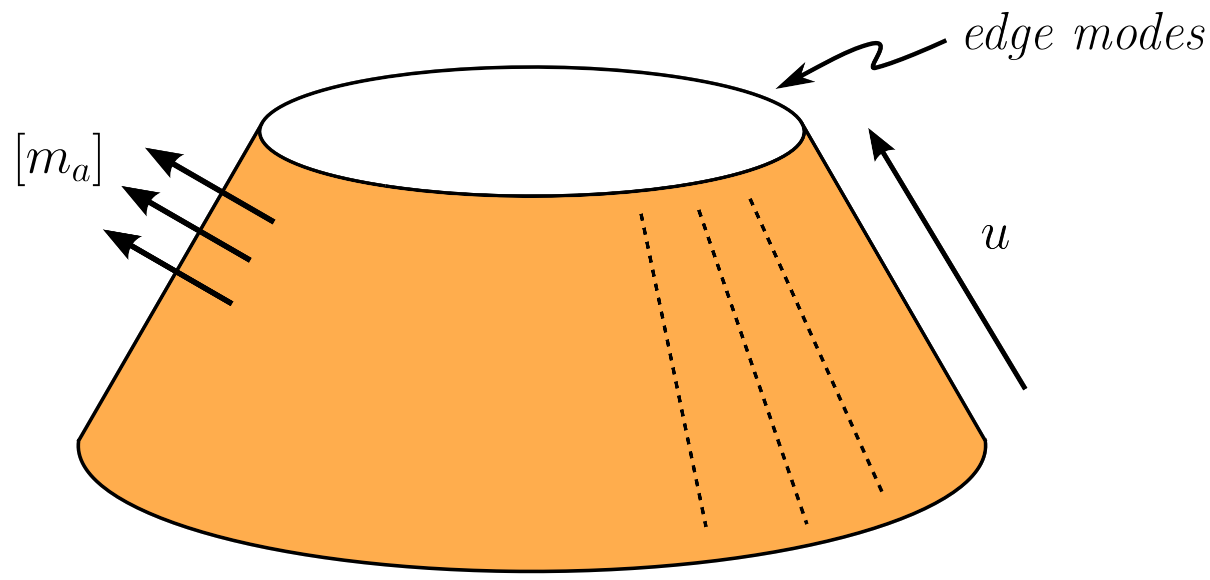


Emerging field of quasi-local observables in LQG: [Riello, Pranzetti, Dittrich, Freidel, Bodendorfer, ww, Corichi,...]

Subsystems bounded by null surfaces

Choice here: identify gravitational subsystems with compact regions of spacetime that are bounded by null surfaces.

- Induced metric: $\varphi^*_{\mathcal{N}} g_{ab} = 2m_{(a}\bar{m}_{b)}$.
- free data: $J = [m_a] : m_a \sim \mathrm{e}^{\mathrm{i}\phi+\lambda} m_a$.
- gauge conditions: non-affinity $\boldsymbol{\kappa}$, and choice of foliation of the null hypersurface (i.e. a choice of time variable *u*).
- free corner data (edge modes): conformal factor, out and ingoing expansion, outgoing shear, plus one additional spin coefficient (NP scalar $\boldsymbol{\tau}$).



[Bondi, Sachs, Penrose, Winicour, Goldberg, Robinson, Soteriou, Reisenberger, …]

Bulk plus boundary fields

- Bulk fields: tetrad, Plebański 2-form, self-dual connection,

$$\Sigma_{AB} = -\frac{1}{2} e_{AC'} \wedge e_B{}^{C'},$$
$$F^A{}_B = \mathrm{d}A^A{}_B + A^A{}_C \wedge A^C{}_B.$$

- Boundary fields: dyadic one-forms, spin frame,

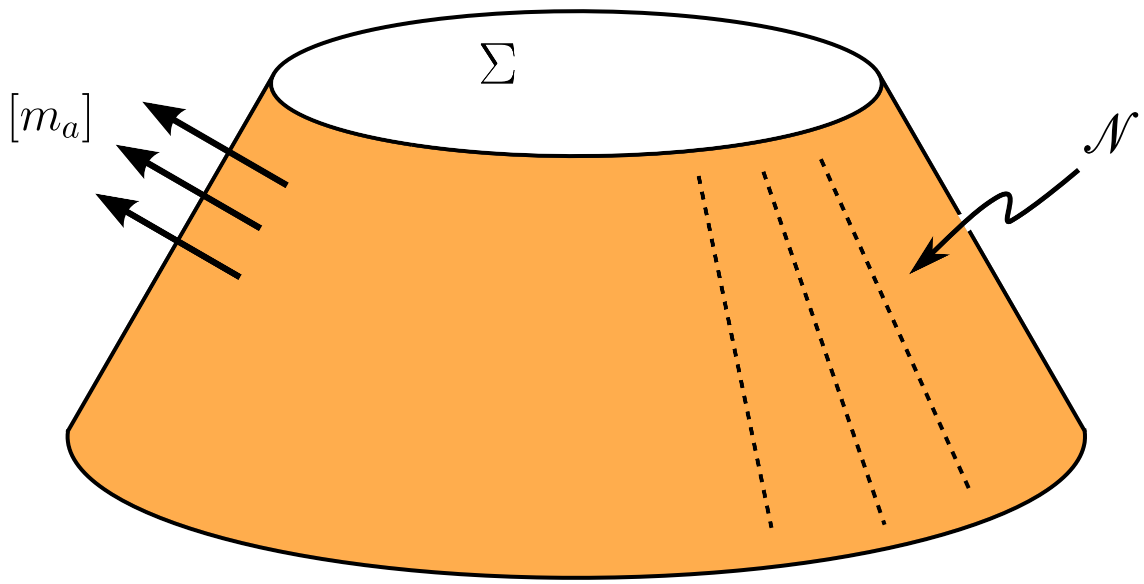
$$\varphi^*_{\mathcal{N}} e^{AA'} = -\mathrm{i}\ell^A \bar{\ell}^{A'} \mathrm{d}u + \mathrm{i}k^A \bar{\ell}^{A'} m + \mathrm{i}\ell^A \bar{k}^{A'} \bar{m},$$
$$(k^A, \ell^A) : k_A \ell^A = 1.$$

- Covariant derivative:

$$D\ell^A = \mathrm{d}\ell^A + \varphi^*_{\mathcal{N}}[A^A{}_B]\ell^B.$$

- Conformal boundary conditions:

$$\delta[m] \propto m \Leftrightarrow m \wedge \delta[m] = 0,$$
$$\delta[u] = 0.$$



Bulk plus boundary action

Variation of the coupled bulk plus boundary action for given boundary conditions returns the bulk plus boundary equations of motion.

- Einstein equations in the bulk.
- Boundary equations of motion: *(i) gluing conditions and (ii) gauge fixing for u.*

$$\textit{(i)} \quad \varphi^*_{\mathcal{N}}\Sigma_{AB} = \big(\ell_{(A}\mathrm{d}u - k_{(A}m\big)\wedge\ell_{B)}\bar{m},$$

$$\textit{(ii)} \quad m\wedge\bar{m}\wedge k_A D\ell^A - \mathrm{cc.} = 0 \Leftrightarrow \textit{u is an affine parameter.}$$

$$S[A,e|(k,\ell),m] = \frac{\mathrm{i}}{8\pi G}\Bigg[\int_{\mathscr{M}}\Sigma_{AB}\wedge F^{AB} + \int_{\mathcal{N}}\big(\ell_A\mathrm{d}u - k_A m\big)\wedge D(\ell^A\bar{m})\Bigg] + \mathrm{cc.}$$

Symplectic structure, gauge symmetries

- Resulting symplectic structure has bulk plus boundary terms:

$$\Theta_\Sigma = \frac{\mathrm{i}}{8\pi G}\Big[\int_\Sigma \Sigma_{AB}\wedge \mathbb{d}A^{AB} - \oint_{\partial\Sigma}(\ell_A \mathrm{d}u - k_A m)\wedge\mathbb{d}(\ell^A\bar{m})\Big] + \mathrm{cc.}$$

- Gauge symmetries are null directions: $\Omega_\Sigma = \mathbb{d}\Theta_\Sigma : \Omega_\Sigma(\delta_{\mathrm{gauge}},\cdot) = 0.$
  - Diffeomorphisms that vanish at the boundary $\delta_\xi[\cdot] = \mathscr{L}_\xi[\cdot] : \xi^a\big|_{\mathcal{N}} = 0.$
  - U(1) flag rotations $\delta_\phi\big[\ell^A, k^A, m\big] = \frac{\phi}{2\mathrm{i}}\big[\ell^A, -k^A, 2m_a\big].$
  - Simultaneous $SL(2,\mathbb{C})$ gauge transformations of bulk plus boundary fields.

Tangent vectors to covariant phase space

Tangent vectors $\delta \in T\mathscr{P}_\Sigma$ to the covariant phase space are linearised solutions of the bulk plus boundary field equations *that must also satisfy* boundary conditions $m \wedge \delta m = 0,\ \delta u = 0.$

- A generic diffeomorphism will not satisfy this condition, e.g. $m \wedge \mathscr{L}_\xi m \neq 0.$
- Define field variation δ_ξ via a projection $\delta_\xi[\cdot] = \mathrm{pr}(\mathscr{L}_\xi[\cdot]).$

$$\text{for all } \delta \in T\mathscr{P}_\Sigma \ : \ \Omega_\Sigma(\delta_\xi, \delta) = \Omega_\Sigma(\mathscr{L}_\xi, \delta).$$

Quasi-local charges

- Charges obtained by integrating Hamilton's equations $\Omega(X_O, \delta) = -\delta[O]$.

- Simplest Dirac observables: generators of boundary dilations and tangential diffeomorphisms.
  - boundary dilations: $\delta_\lambda\big[\ell^A, k^A, m\big] = \frac{\lambda}{2}\big[\ell^A, -k^A, 0\big], \quad m \wedge \bar{m} \wedge \mathrm{d}\lambda = 0.$
  - tangential bulk plus boundary diffeomorphisms: $\xi^a : \xi^a|_{\partial\Sigma} \in T\Sigma$.

boundary dilations	$K_\lambda[\mathscr{C}] = \frac{\mathrm{i}}{16\pi G}\oint_{\mathscr{C}} \lambda(\eta_A \ell^A - \mathrm{cc.}),$
tangential diffeos	$J_\xi[\mathscr{C}] = \frac{\mathrm{i}}{8\pi G}\oint_{\mathscr{C}} \xi^a\big(\eta_A D_a \ell^A - \mathrm{cc.}\big).$

**where we introduced the abbreviation* $\eta_A = (\ell_A \mathrm{d}u - k_A m) \wedge \bar{m}$.

Special vector fields

It is instructive to evaluate the charge on area-preserving vector fields on u=const. cross sections of the null boundary $\xi_f^a = 2\mathrm{i} m^{[a}\bar{m}^{b]} D_b f \in T\mathscr{C}_u$.

$$J_{\xi_f}[\mathscr{C}] = -\frac{\mathrm{i}}{4\pi G}\oint_{\mathscr{C}} m \wedge \bar{m}\, f\, \mathfrak{Im}\Big[\Psi_2 - \bar{\sigma}_{(k)}\sigma_{(\ell)}\Big].$$

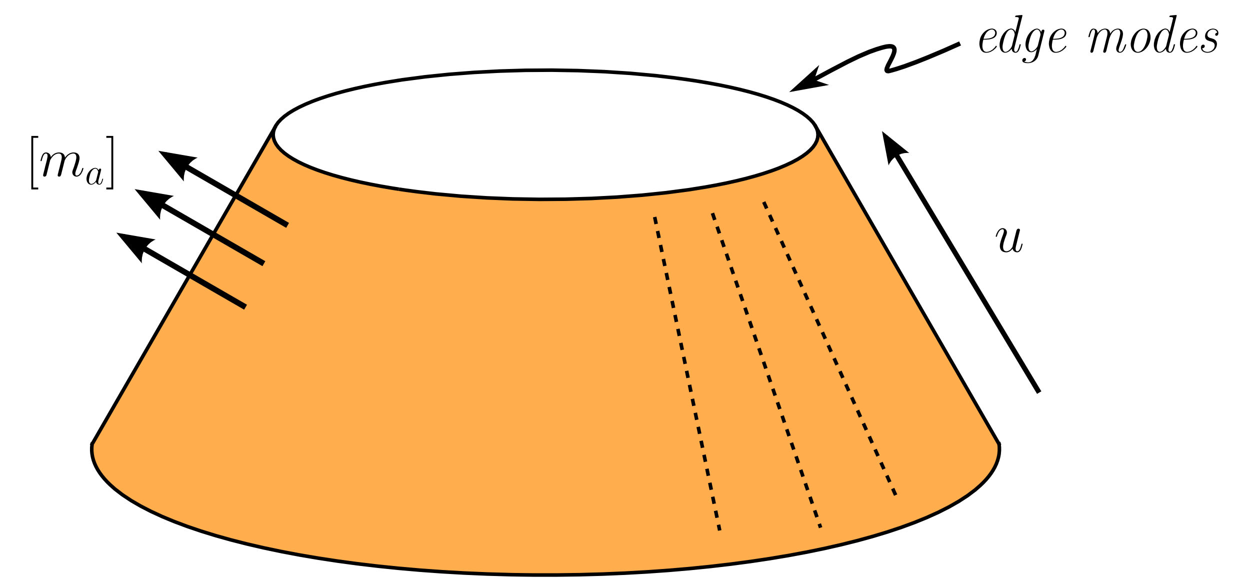
*note: $\Psi_2 : \varphi^*_{\mathscr{C}_u}(F_{AB}k^A\ell^B) = -\Psi_2\varphi^*_{\mathscr{C}_u}(m \wedge \bar{m})$.

Quasi-local Hamiltonian

- Smoothly extend the null flag ℓ^A inside, define the null vector

$$\xi^a = \mathrm{i}\,\ell^A\bar{\ell}^{A'}e_{AA'}{}^a.$$

- Define the field variation δ_H (its projection onto $\mathscr{P}_\Sigma$ is Hamiltonian).



boundary translation

$$\begin{cases} \delta_H[\ell^A] := \xi^a D_a \ell^A + \frac{1}{4}\vartheta_{(\ell)}, \\ \delta_H[k^A] := \xi^a D_a k^A - \frac{1}{4}\vartheta_{(\ell)}, \\ \delta_H[m] := \xi \lrcorner \mathrm{d}m. \end{cases}$$

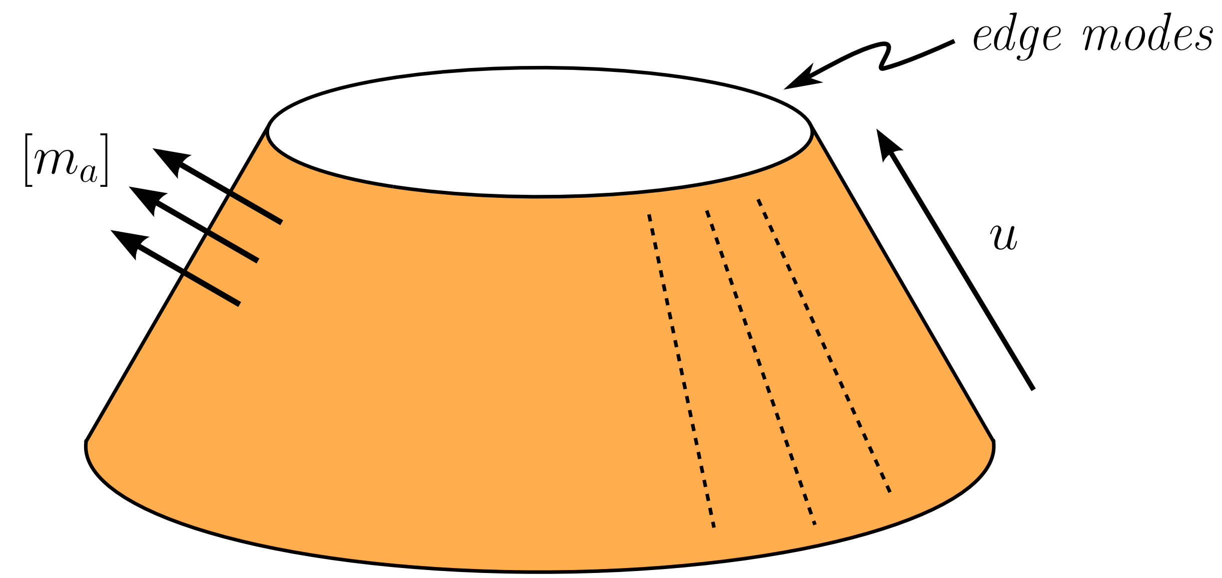
bulk translation

$$\begin{cases} \delta_H[e^{AA'}] := \mathrm{i}\nabla(\ell^A\bar{\ell}^{A'}), \\ \delta_H[A^A{}_B] := \xi \lrcorner F^A{}_B. \end{cases}$$

Expansion as quasi-local Hamiltonian

- The quasi-local Hamiltonian is given by the expansion (plus an irrelevant constant).

$$H_\Sigma = \frac{\mathrm{i}}{8\pi G}\oint_{\partial\Sigma} \vartheta_{(\ell)} m \wedge \bar{m}.$$



- The *u-translations* are integrable on $\mathscr{P}_\Sigma$:

$$\delta[H_\Sigma] = \Omega_\Sigma(\delta, \delta_H) - \frac{\mathrm{i}}{8\pi G}\oint_{\partial\Sigma}\big[\bar{\sigma}_{(\ell)} \underbrace{m \wedge \delta m}_{\text{vanishes on } \mathscr{P}_\Sigma} - \mathrm{cc.}\big].$$

What does this all

have to do with LQG?

Action with Barbero-Immirzi term

- In the Palatini formalism, the addition of a term $\frac{1}{\gamma}\varepsilon^{abcd}R_{abcd}$ does not change the equations of motion in the interior. *But it does change the boundary field theory.*

- Bulk plus boundary action

$$S[A,e|(k,\ell),m] = \frac{\mathrm{i}}{8\pi G}\frac{\gamma+\mathrm{i}}{\gamma}\Big[\int_{\mathscr{M}}\Sigma_{AB}\wedge F^{AB} + \int_{\mathscr{N}}(\ell_A\mathrm{d}u - k_A m)\wedge D(\ell^A\bar{m})\Big] + \mathrm{cc.}$$

- Bulk plus boundary symplectic structure

$$\Theta_\Sigma = \frac{\mathrm{i}}{8\pi G}\frac{\gamma+\mathrm{i}}{\gamma}\Big[\int_{\Sigma}\Sigma_{AB}\wedge \mathbb{d}A^{AB} + \oint_{\mathscr{C}} k_A m\wedge \mathbb{d}(\ell^A\bar{m})\Big] + \mathrm{cc.}$$

Boundary symplectic structure

- Introduce fiducial dyadic basis:

$$m \wedge \delta m = 0 : m = e^{\omega} m_o,$$
$$d^2\Omega_o = \mathrm{i} m_o \wedge \bar{m}_o.$$

- Canonical conjugate boundary variables:

$$\tilde{\ell}^A = e^{\bar{\omega}} \ell^A,$$
$$\tilde{\pi}_A = \frac{1}{8\pi G} \frac{\gamma + \mathrm{i}}{\gamma} k_A e^{\omega} d^2\Omega_o.$$

Boundary reality condition

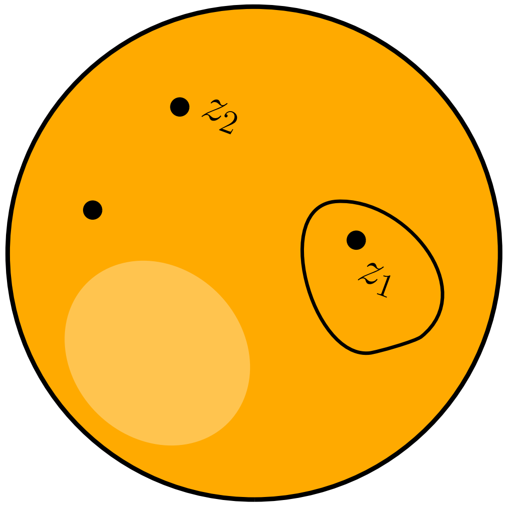
- Generators of complexified U(1) transformations

null dilatations	$K_\lambda[\mathscr{C}] = \frac{1}{2}\oint_{\mathscr{C}} \lambda\big(\tilde{\pi}_A\tilde{\ell}^A + \text{cc.}\big),$
U(1) flag rotations	$L_\phi[\mathscr{C}] = \frac{1}{2\mathrm{i}}\oint_{\mathscr{C}} \phi\big(\tilde{\pi}_A\tilde{\ell}^A - \text{cc.}\big).$

- Reality conditions on the area two-form implies a constraint:

$$K - \gamma L = 0.$$

LQG boundary states



- Consider a Schrödinger quantisation of the boundary phase space.
  - States are functionals on configuration space.
  - LQG assumption: geometry excited only in a superposition of punctures.

$$\Psi_f[\ell] = f\big(\ell(z_1), \dots \ell(z_N)\big).$$

- Momentum smeared around a puncture acts as an ordinary derivative

$$\int_{U_i} \big(\hat{\pi}_A \Psi_f\big)[\ell] = -\mathrm{i}\frac{\partial}{\partial \ell_i^A} f\big(\ell(z_1), \dots \ell(z_N)\big),$$

$$\int_{U_i} \big(\hat{\pi}^\dagger_{A'} \Psi_f\big)[\ell] = -\mathrm{i}\frac{\partial}{\partial \bar{\ell}_i^{A'}} f\big(\ell(z_1), \dots \ell(z_N)\big).$$

- Irreducible unitary representations of *SL*(2,$\mathbb{C}$) labelled by homogenous functions

$$\big(\hat{K} + \mathrm{i}\hat{L}\big) f(\ell) = (\rho + \mathrm{i}k) f(\ell), \quad (\rho, k) \in \mathbb{R} \times \mathbb{Z}/2$$

LQG quantisation of area

- $K = \gamma L$ condition implies $\rho = \gamma j$ restriction on allowed *SL*(2,ℂ) representations.
- Each puncture carries discrete eigenvalues of area-flux:

$$\text{spectrum of area flux} = 8\pi\,\hbar G\gamma \times \mathbb{Z}/2.$$

- Typical wave functions are of the form:

$$f(\ell) = \big(-\mathrm{i}X_{AA'}\ell^A\bar{\ell}^{A'}\big)^{\mathrm{i}\gamma j - j - 1}\,\bar{\ell}^{A'_1}\ldots\bar{\ell}^{A'_{2j}}.$$

- These are the same type of wave functions that appear in LQG [Freidel, Speziale, Bianchi, Donà, Dupuis, Levine, Girelli, ww].

Summary and conclusion

Loop gravity spinor representation can be understood from the quantisation of gravitational boundary (edge) modes that arise at the boundary of causal regions.

In addition, quantum discreteness of area also compatible with conventional Fock space quantization in the continuum. [1,2,3]:

- Entire construction is manifestly Lorentz covariant.
- When applied to three dimensions, new connection between CFTs and gravity [3].
- Fundamental boundary variables are the same that underpin twistor string theory and amplitudes [Arkani-Hamed, Cachazo, Penrose, Mason, Skinner, Adamo].

[1] ww, New boundary variables for classical and quantum gravity […], **Class. Quantum Grav. 34 (2017)**]

[2] ww, Fock representation of gravitational boundary modes […], **Annales Henri Poincaré 18 (2017)**]

[3] ww, Conformal boundary conditions, loop gravity and the continuum, **JHEP (2018):89**]