Holographic Special Relativity:
Observer Space from Conformal Geometry

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(Special) relativity is *not* fundamentally about spacetime. It is about how different observers' viewpoints relate to each other.

Spacetime is one way to understand these viewpoints:

- Lorentzian manifold M
- An *observer* is a unit future-directed timelike tangent vector

but not the only way ....

Motivations

- (A)dS/CFT
- Shape dynamics (a conjecture)
- Observer space geometries ...

"Universal" geometry for theories of space and time.

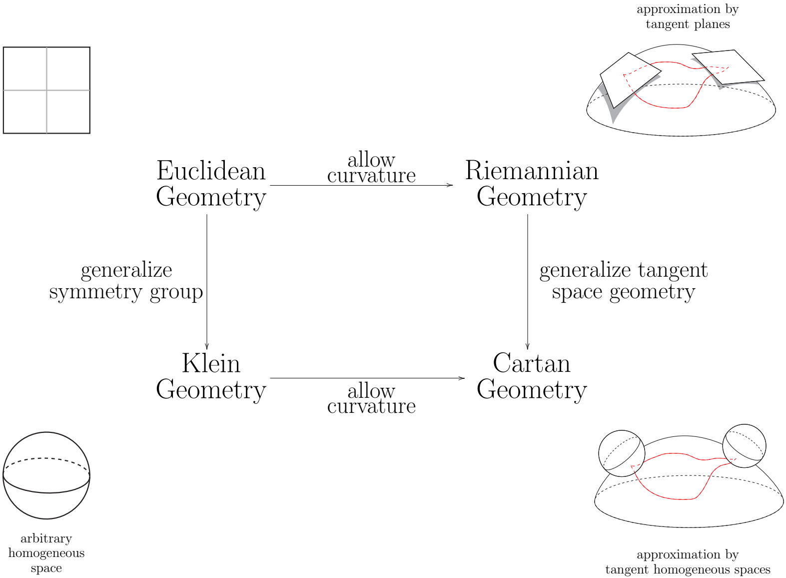
- Relates covariant and canonical pictures of gravity
  [S. Gielen, DKW, 1111.7195, 1206.0658; DKW, 1310.1088]
- "Einstein equations on observer space"

$$\Longrightarrow \quad \begin{cases} \text{Spacetime as a quotient of observer space} \\ \text{Einstein equations on reconstructed spacetime} \end{cases}$$

  [SG, DKW, 1210.0019]
- More general than Lorentzian spacetime:
  - Preferred-foliation (e.g. Hořava grav.) [S. Gielen, 1301.1692]
  - Finsler spacetime [M. Hohmann, 1304.5430]
  - No spacetime at all (e.g. relative locality)
  - From conformal geometry [now!]

Defined in terms of Cartan geometry....

Cartan geometry



(Adapted from diagram by R.W. Sharpe.)

De Sitter space is the homogeneous space $S^{3,1} \cong \mathrm{SO}_o(4,1)/\mathrm{SO}(4,1)$.

$\mathrm{SO}_o(4,1)$ acts transitively on observers, with stabilizer $\mathrm{SO}(3)$, so the observer space (unit future tangent bundle) of de Sitter space is $\mathrm{SO}_o(4,1)/\mathrm{SO}(3)$.

Definition: An **observer space geometry** is a Cartan geometry modeled on de Sitter observer space $\mathrm{SO}_o(4,1)/\mathrm{SO}(3)$.

(...or Minkowski or AdS analogs, ... also in other dimensions)

Basic example: The unit future tangent bundle of a Lorentzian manifold has a canonical observer space geometry.

Mathematical punchline of this talk:

Theorem:

1. An nd conformal geometry *canonically* determines a $(2n+1)$d observer space geometry.
2. If the conformal geometry is the standard n-sphere, the observer space geometry is the observer space of $n+1$ de Sitter spacetime.

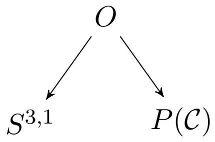
$\Longrightarrow$ de Sitter special relativity without de Sitter spacetime.
$\Longrightarrow$ possibility for observer space dynamics from conformal space, rather than spacetime.

I will stick to $n = 3$, and talk about part 2 until the end, i.e. how to construct the observer space of de Sitter spacetime from conformal space...

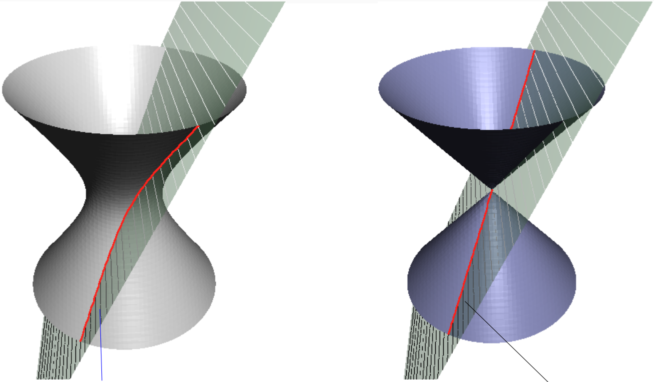
$\mathbb{R}^{4,1}$ — "ambient space"

All of the spaces we need inherit an action of $G := \mathrm{SO}_o(4,1)$ from the action of $\mathrm{SO}_o(4,1)$ on $\mathbb{R}^{4,1}$:

- de Sitter spacetime $S^{3,1}$ (unit spacelike pseudosphere)
- hyperbolic space H^4 (unit future-timelike pseudosphere)
- ambient future/past light cones $\mathcal{C}^+$, $\mathcal{C}^-$
- conformal 3-sphere $P(\mathcal{C})$ (projective lightcone in $\mathbb{R}^{4,1}$)
- observer space $O \cong \{(x,u) \in S^{3,1} \times \mathrm{H}^4 : \eta(x,u) = 0\}$
- inertial observer space $\overline{O}$ (space of timelike geodesics)



Mapping observer space to conformal space



2d subspace $[x, u] \subset \mathbb{R}^{4,1}$ containing the observer's geodesic.

same 2d subspace contains a pair of lightrays

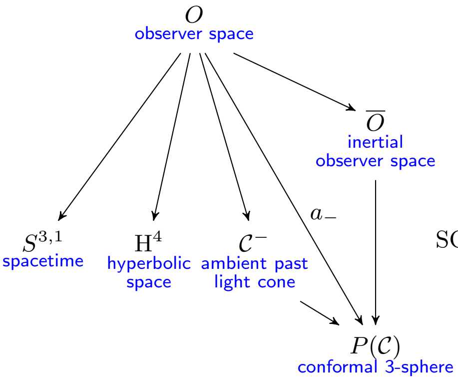
$$[x \pm u] \in P(\mathcal{C})$$

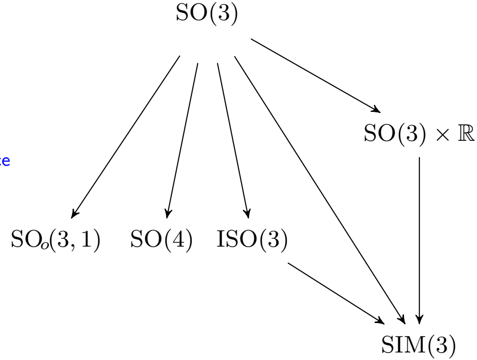
An observer $(x, u) \in O \subset S^{3,1} \times \mathrm{H}^4$ has:
asymptotic future $[x + u]$ and **asymptotic past** $[x - u]$

(notation: $[v, \ldots] = \mathrm{span}\{v, \ldots\}$)

$\mathrm{SO}_o(4,1)$-spaces and equivariant maps

... and stabilizer subgroups





Observer space geometry from conformal space?

OK, fine, so conformal space is a quotient of observer space, *but...*

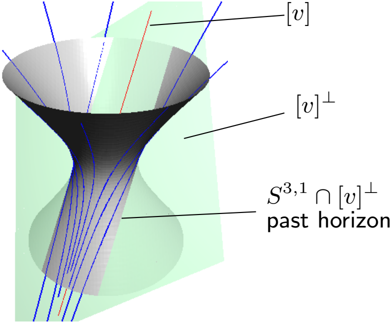
Suppose we don't start out with spacetime.

Can we construct the de Sitter observer space O just from the conformal sphere? ...

And then, can we generalize this construction to other conformal geometries?

For $[v] \in P(\mathcal{C})$, what are *all* observers (x, u) with $[x - u] = [v]$?
Answer:

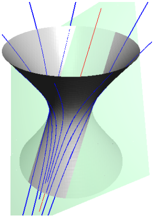
All observers who share the same *past cosmological horizon.*



This whole *field of observers* corresponds to a point in $P(\mathcal{C})$.

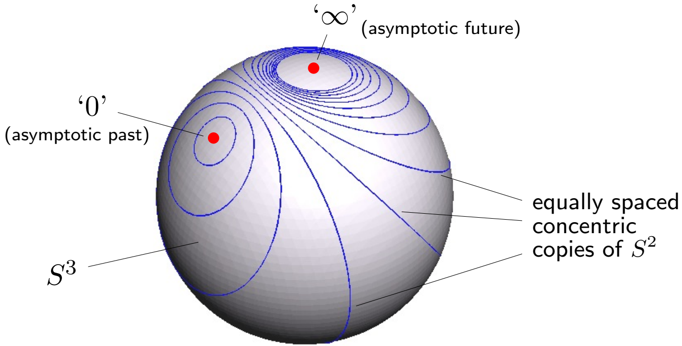
Theorem: If you know where you came from and where you're going, then you know who you are.

(More precisely: An *inertial observer* is uniquely determined by two distinct points in $P(\mathcal{C})$, the 'asymptotic past' and 'asymptotic future'.)



Theorem: The more you can see of the past, the older you are.

(More precisely: The time along an inertial observer's worldline is uniquely determined by a 2-sphere in $P(\mathcal{C})$, ...)



The conformal compactification of $\mathbb{R}^3$ is $S^3 \cong P(\mathcal{C})$

Conversely, choosing '0', '∞', and the 'unit sphere' S^2 in $P(\mathcal{C})$ makes $P(\mathcal{C}) - \{\infty\}$ into a Euclidean vector space $\cong \mathbb{R}^3$.

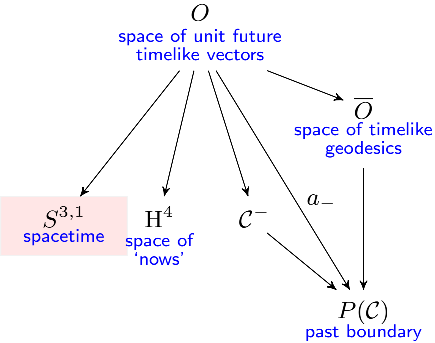
Theorem:

$\{\text{observers in } S^{3,1}\} \cong \{\text{Euclidean de-compactifications of } P(\mathcal{C})\}$

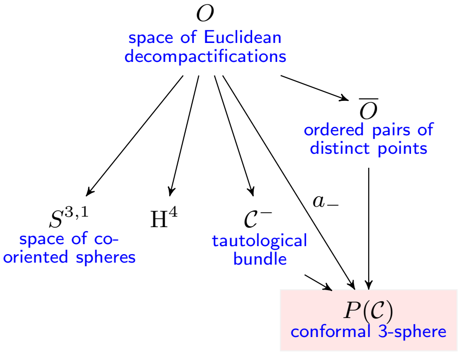
This isomorphism is *canonical* and $\mathrm{SO}_o(4,1)$-equivariant.

Continuing in this way, we get...

Spacetime picture:



Conformal picture:



So far, an observer is very 'nonlocal': need *two* points in $P(\mathcal{C})$ plus a sphere.

Fortunately, the Euclidean decompactification is determined entirely by local data on $P(\mathcal{C})$:

Theorem: The observer space of de Sitter spacetime is isomorphic as a G-space to the space of all transverse 3-planes in the tautological bundle over the conformal 3-sphere.

We can now write the theorem from the beginning more precisely:

Theorem:

1. A conformal geometry *canonically* determines an observer space geometry on the space of transverse 3-planes in the tautological bundle.
2. For the $P(\mathcal{C})$, this coincides with the observer space associated to $S^{3,1}$.

We can also write *integrability conditions* that allow reconstruction of conformal space from a general observer space geometry, analogous to the conditions for reconstruction of spacetime in [1210.0019].

What do Cartan geometries actually look like? ...

Cartan geometry modeled on G/H involves breaking G symmetry to H to split a connection into pieces....

Familiar example: MacDowell–Mansouri gravity:

$$G = \mathrm{SO}_o(4,1) \qquad H = \mathrm{SO}_o(3,1)$$

As reps of $\mathrm{SO}(3,1)$:

$$\begin{array}{cccccc} & \mathfrak{so}(4,1) & \cong & \mathfrak{so}(3,1) & \oplus & \mathbb{R}^{3,1} \\ \Longrightarrow & A & = & \omega & + & e \\ & & & \text{spin conn.} & & \text{coframe} \end{array}$$

[gr-qc/0611154]

Cartan geometry

Observer space geometry is modeled on $\mathrm{SO}_o(4,1)/\mathrm{SO}(3)$, so [1210.0019]

$$\mathfrak{so}(4,1) = \mathfrak{so}(3) \oplus (\mathbb{R}^3_{\mathrm{b}} \oplus \mathbb{R}^3_{\mathrm{t}} \oplus \mathbb{R})$$

rotations — boosts — spatial translations — time translations

On the other hand, the dual 'holographic' picture of observers suggests a different splitting:

$$\mathfrak{so}(4,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$$

translations of asymptotic past — rotations and time translations — translations of asymptotic future

This splitting is a standard tool in *conformal* Cartan geometry. Here we interpret it terms of observer space geometry.

How are the two decompositions (and their geometric interpretations) related?

$\mathfrak{so}(4,1) = \mathfrak{so}(3) \oplus (\mathbb{R}^3_{\mathrm{b}} \oplus \mathbb{R}^3_{\mathrm{t}} \oplus \mathbb{R})$

$\mathfrak{so}(4,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$

- $\mathfrak{so}(3)$ is the stabilizer of $(x,u) \in O$
- $\mathfrak{g}_0 = \mathfrak{so}(3) \oplus \mathbb{R}$ (stabilizer of inertial observer)
- $\mathfrak{g}_{\pm 1} = \{(B,T) \in \mathbb{R}^3_{\mathrm{b}} \oplus \mathbb{R}^3_{\mathrm{t}} : B = \pm T\}$
- $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ is the stabilizer of $[x-u] \in P(\mathcal{C})$

S. Gryb and F. Mercati, 1209.4858:

- $2+1$ split of 3d gravity (Chern–Simons with $G = \mathrm{SO}_o(3,1)$)
- write the fields according to the split

$$\mathfrak{so}(3,1) \cong \mathfrak{so}(2) \oplus (\mathbb{R}^2_{\mathrm{b}} \oplus \mathbb{R}^2_{\mathrm{t}} \oplus \mathbb{R})$$

- take linear combinations of fields to reorganize them according to

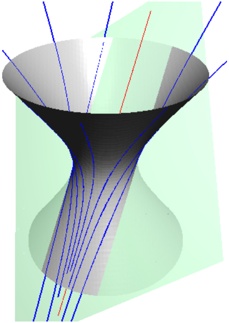
$$\mathfrak{so}(3,1) \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$$

- interpret these fields as living on a 2-sphere, thus giving a conformal Cartan connection.
- show the result is *equivalent to shape dynamics* for 3d GR!

But the *geometric meaning* of this reorganization of fields is explained by observer space...

In particular:

The 2d conformal space in this theory is evidently *not* "space", but some *nonlocal* quotient of observer space. For 3d de Sitter spacetime, a *point* in this conformal space looks like this in spacetime:



So, is shape dynamics really is a theory of dynamical "spatial" conformal geometry?

Conclusions

- New examples of observer space geometries! Questions:
  - Dynamics for physically realistic ‘holographic general relativity’?
  - Is *shape dynamics* exactly that?
- Challenge: Geometric interpretation of shape dynamics?
  - Is this a special trick of 3d gravity?
  - Try doing it for 4d starting from [S. Gielen, DKW, 1111.7195], keeping all of the Cartan geometric aspects explicit.
  - Is the nonlocal nature of conformal ‘space’ responsible for the nonlocal Hamiltonian in shape dynamics?
  - Is the ‘linking theory’ [1101.5974] best framed in terms of observer space?

$$\begin{array}{ccc} & O & \\ \swarrow & & \searrow \\ S^{3,1} & & P(\mathcal{C}) \end{array}$$