Basket Implied Volatility from Geodesics

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Abstract. This paper presents a deterministic numerical method to calculate the implied volatility of an option based on multiple assets using a stochastic volatility model. The approach uses Varadhan asymptotics for the diffusion kernel and involves the constrained minimization of a numerically computed geodesic length. We produce implied volatilities for baskets with 10 – 50 elements using the SABR stochastic volatility model and compare results with those obtained from Monte Carlo simulation. We find that the approach is significantly faster than Monte Carlo. We also present results using a domain decomposition preconditioner (additive Schwarz type) which significantly improves performance for the constrained minimization.

1. Introduction

Options based on multiple assets which do not follow log-normal processes are of theoretical and practical interest. The size of the problem frequently precludes using lattice-based methods for treating partial differential equations, including finite difference methods and binomial trees, and usually requires Monte Carlo simulation for solution. However, the slow convergence of Monte Carlo techniques can be a drawback.

Several approaches to basket option pricing exist. A partial list includes Monte Carlo [7, 19], multinomial trees [21], universal bounds [17], and analytical approximations [13]. Computational expense is a significant hurdle in many approaches to basket option pricing. Incorporating stochastic volatility adds more computational expense and complication.

Recently several researchers have examined stochastic volatility using methods based on differential geometry. Using the heat kernel expansion, stochastic volatility models in single asset cases have been explored in [11, 15, 2, 9, 10]. Heat kernel asymptotics were recently used for exploring the multidimensional Black-Scholes formula [12].

M. Avellaneda et al. [3] used the method of steepest descent for diffusion kernels in order to produce the local volatility function of a basket and construct the implied volatility of an option on a basket. The result was an closed form analytic formula which closely matched market quotes. M. Avellaneda has also applied Varadhan asymptotics and the method of steepest descent to a stochastic volatility model for a single asset case[2]. In this work, we present a general numerical method based on the approach of M. Avellaneda in [2] for multiple asset stochastic volatility models. We find significant computational advantages over Monte Carlo using this approach. We explore the simplest stochastic volatility model, SABR [14], in our numerical experiments; however, the approach is not limited to just SABR.

We also explore domain decomposition preconditioning as part of the numerical constrained minimization algorithm. Additive Schwarz type domain decomposition preconditioning is frequently used in solving all types of partial differential equations [18, 20, 1]. We find that additive Schwarz type preconditioning significantly improves the constrained minimization performance that is crucial for dealing with large numbers of assets. We present performance results with and without using additive Schwarz type preconditioning.

The structure of this paper is as follows. In Sec. 2 we describe the stochastic volatility model explored in the numerical tests. The implied volatility expression for baskets with small time expiries as derived by M. Avellaneda is outlined and the numerical approach for solving the expression is described. In Sec. 3, we compare results from the implied volatility expression with results obtained using Monte Carlo for baskets of 10, 20, and 50 assets with and without domain decomposition type preconditioning. We then use domain decomposition type preconditioning and the implied volatility expression for small expiry times to compare the implied volatility at multiple strike prices and expiry times with that obtained using Monte Carlo for baskets of 10, 25, and 50 assets. Finally, in Sec. 4 we summarize the results.

2. Numerical Approach

We examine a basket B of n assets with time-dependent prices $f_k = f_k(t)$ and constant weights w_k:

$$B = \sum_{k=1}^{n} w_k f_k. \tag{1}$$

We adopt the following multiple asset SABR model based on the stochastic differential equations (SDEs) given in [14]:

$$df_k = a_k f_k^{\beta} dW_k \tag{2}$$

$$da_k = \nu_k a_k dZ_k, \tag{3}$$

where f_k is the price of the k-th asset, a_k is the volatility, ν_k is the non-stochastic diffusion coefficient of the volatility, β is a constant, and W_k and Z_k are correlated Brownian processes with correlation matrix

$$\mathbf{P} = (\rho_{ij}),\ 1 \le i \le 2n,\ 1 \le j \le 2n$$

and elements $\rho_{ij} \in [-1, 1]$.

The implied volatility for the stochastic volatility model is the volatility in the Black-Scholes European call option formula which gives the same asset price. An asymptotic expression obtained using singular perturbation techniques which gives the single asset implied volatility for the SABR model is derived in [14]. M. Avellaneda [2] used Varadhan asymptotics for the heat kernel and the steepest descent approximation to obtain a general expression capable of giving the implied volatility for the single or multiple asset SABR models. In the single asset case, both approaches produce equivalent results for small expiry times.

The expression derived by M. Avellaneda is subject to the limits of Varadhan asymptotics and is consequently only valid for small expiry times. Similar expressions for the implied volatility specific to the single asset case which do not use steepest descent have been derived in at least two other places [9, 15]. The Avellaneda expression for the implied volatility of a basket with n assets is:

$$\sigma_{\mathrm{BS}} = \frac{\ln\left(B/B_0\right)}{\min\left(L(x,y) | y : \sum_{k=1}^{n} w_k f_k = B\right)} \tag{4}$$

$$x = (a_1^0, f_1^0, \ldots, a_n^0, f_n^0)$$
$$y = (a_1, f_1, \ldots, a_n, f_n),$$

where σ_{BS} is the implied volatility, B_0 is the basket spot price, B is the basket strike with expiry at time T, x is the spot configuration of the basket elements, a_k^0 and f_k^0, y is the expiry configuration of the basket elements, a_k and f_k, and $L(x,y)$ is the geodesic length between x and y. Thus we vary y to minimize the geodesic length between x and y subject to the constraint that $\sum_{k=1}^n w_k f_k = B$ in order to find the implied volatility. The geodesic length $L(x,y)$ is given by

$$L(x,y) = \inf_{\gamma(0)=x,\gamma(1)=y} \int_0^1 \sqrt{\sum_{ij=1}^{2n} g_{ij} \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}} ds, \tag{5}$$

where $\gamma(s)$ is the geodesic with endpoints at x and y and g_{ij} is the metric. The metric and its inverse are given by

$$g_{ij} = \frac{\rho^{ij}}{\sigma_i \sigma_j} \tag{6}$$
$$g^{ij} = \rho_{ij}\sigma_i\sigma_j, \tag{7}$$

where ρ_{ij} is the correlation matrix element, ρ^{ij} is the inverse correlation matrix element, and the quantities σ_i and σ_j are the diffusion coefficients appearing in the SDE of the particular stochastic volatility model chosen for study. For Eqn. (2), $\sigma_k = a_k f_k^\beta$ and for Eqn. (3), $\sigma_k = \nu_k a_k$.

The crucial component of solving Eqn. (4) is finding and minimizing the geodesic distance in arbitrary dimensions. There are multiple ways to find the geodesic $\gamma(s)$ between x and y given a metric g_{ij}. A straightforward way is to solve the geodesic equation:

$$\frac{\partial^2\gamma^i}{\partial s^2} + \Gamma^i_{jk}\frac{\partial\gamma^j}{\partial s}\frac{\partial\gamma^k}{\partial s} = 0, \tag{8}$$

where Γ^i_{jk} are the connection coefficients. In our experiments found that solving Eqn. (8) directly from the metric is generally too time consuming for large numbers of assets, especially when combined with the minimization required for Eqn. (4). Instead of solving Eqn. (8) and then computing the integral in Eqn. (5), we enforce the extremal length condition of a geodesic by minimizing the expression

$$l(\mathbf{h}) = \int_0^1 \sqrt{\sum_{ij=1}^{2n} g_{ij}\frac{dh^i}{ds}\frac{dh^j}{ds}} ds. \tag{9}$$

for a generic family of curves $\mathbf{h}$ where $h(0) = x$ and $h(1) = y$, x being the spot configuration of the basket elements and y being the expiry configuration of the basket elements. By supplying a sufficiently generic family of curves and optimizing the parameters until $l(\mathbf{h})$ in Eqn. (9) is minimized, we find an approximation to the geodesic, $\gamma(s)$.

The generic family of curves we used to minimize Eqn. (9) is given by the differential equation

$$\frac{\partial^2 h^i}{\partial s^2} + c^i\frac{\partial h^i}{\partial s} + d^i = 0, \tag{10}$$

with parameters c^i and d^i. We vary the parameters c^i and d^i in order to minimize $l(\mathbf{h})$ in Eqn. (9). We use boundary conditions for Eqn. (10) that are consistent with Eqn. (5):

$h(0) = x$ and $h(1) = y$. The family of curves specified by Eqn. (10) was found empirically; other choices exist.

The constrained minimization required by Eqn. (4) was performed using the optimization utilities provided in the PETSc and TAO toolkits [5, 4, 6, 8]. We used the limited-memory variable metric algorithm. We use the initial volatility states, $(a_k^0)_{k=1\dots n}$ as the initial guess for the final volatility configuration, $(a_k)_{k=1\dots n}$ and the basketstrike price as the initial guess for the final price configuration, $(f_k)_{k=1\dots n}$ for simulations without preconditioning and for the local problem stage of domain decomposition preconditioning. The initial guess for the parameters in Eqn. (10), c^i and d^i, is 1.0. The geodesic $\gamma(s)$ is then found by the process described above and the geodesic length is calculated by performing the integral in Eqn. (5) using 12 point Gaussian quadrature. The gradient of the geodesic length is calculated using a finite difference approximation. This process is repeated until the optimization solve tolerance is met and the implied volatility is calculated from Eqn. (4).

A domain decomposition type preconditioner was used to improve the constrained optimization performance. Domain decomposition methods are well known for problems originating from the discretization of differential equations. The two key components of a domain decomposition preconditioner consist in addressing two questions: first, how to divide a given global problem into smaller local problems which can be solved giving local approximate data; and second, how to gather these local approximate data to create global approximate data which are an approximation to the solution of the global problem.

In the constrained minimization problem considered here, the idea of domain decomposition is applied in a purely algebraic setting: we define the local problem by partitioning the full basket into several local baskets each consisting of a small number of randomly selected assets so that each element of the full basket is located in at least one specific local basket. To define a local basket, we first choose how many local baskets will be used. Let n_B be the number of local baskets, n_l be the number of assets in the l-th local basket B^l, and n be the number of assets in the full basket. We then define local basket index I^l for B^l such that every asset is found in at least one local basket:

$$\bigcup_{l=1}^{n_B} I^l = \{1, 2, \cdots, n\}.$$

The local baskets B^l are defined as

$$B^l = \{(a_k^0, f_k^0) : k \in I^l\},$$

where $l = 1, 2, \cdots, n_B$. The constrained minimization algorithm is then solved for each local basket:

$$\min \left(L(x, y) | y : \sum_{k \in I^l} w_k^l f_k = B \right) \tag{11}$$

$$\begin{aligned} x &= \{(a_k^0, f_k^0) : k \in I^l\} \\ y &= \{(a_k, f_k) : k \in I^l\}, \end{aligned} \tag{12}$$

where B is the full basket strike price and local basket weights w_k^l are scaled from their original values so that they are equal to the sum of the whole basket weights,

$$\sum_{k \in I^l} w_k^l = \sum_{k=1}^{n} w_k.$$

We define $\hat{y}^l$ for each local basket to be of size n

$$\hat{y}_i^l = \begin{cases} y_i^l & \text{if } i \in I^l \\ 0 & \text{otherwise} \end{cases}$$

using the result found from solving Eqn. (11). The results from the various local basket constrained minimizations are then collected into an approximate solution of the full basket problem and used as the initial guess for the full constrained optimization problem:

$$y_{\text{guess}} = \sum_{l=1}^{n_B} \hat{y}^l.$$

In the case where the same asset is located in several local baskets, the y result is averaged over the overlap assets:

$$y_{\text{guess}} = \sum_{l=1}^{n_B} \bar{\hat{y}}^l.$$

This preconditioner significantly reduces the time required for the numerical constrained minimization of Eqn. (4).

To test the results of the method, we compare results with those obtained from Monte Carlo, where Eqns. (2)–(3) are integrated using an implicit Runge-Kutta method [16] for SDEs of Itô type. For single asset tests, we recover the asymptotic expression given by Hagan et al. [14].

3. Results

In this section we present the implied volatility for baskets consisting of 10–50 assets obtained by solving Eqn. (4) and by using Monte Carlo simulation for comparison. We first present results comparing the geodesic approach of Eqn. (4) with the SABR asymptotic expression at multiple expiry times. We then present a comparison of the geodesic approach with and without domain decomposition preconditioning and with Monte Carlo. Preconditioned cases used either five or ten local subdomains. Afterwards we present implied volatility curves for baskets of 10, 25, and 50 assets and compare those to Monte Carlo results at multiple expiry times.

The expression for implied volatility derived by M. Avellaneda, Eqn. (4), produces results which closely match the exact solution for small time expiries and small volatility. For a single-asset example we compare the geodesic approach with the SABR asymptotic expression[14]. Figure 1 compares the asymptotic expression at multiple expiry times against the results found using the geodesic approach. The SABR parameters chosen in this single asset example are similar to those used in the basket examples. In the limit of small time, the implied volatility expression converges to the asymptotic expression. For an expiry time of 0.1 and SABR parameters $a^0 = 0.5, \nu = 2.5, \rho = -0.3$, the error was about 2%. The Avellaneda expression underestimates the actual solution as the expiry time increases.

The domain decomposition preconditioner substantially improves the performance of the geodesic approach as the number of dimensions increases. Table 1 compares the time required for the constrained optimization solution to converge with and without the preconditioner for a wide range of dimensions. For comparison, the time required for Monte Carlo using 20000 samples is also given. The parameters for the elements of the basket (a_k^0, f_k^0, ν_k) were generated randomly with $a_k^0 \in [0.3 \ldots 0.5]$, $f_k^0 \in [45 \ldots 49]$, $\nu_k \in [1.1 \ldots 2.9]$; β was selected to be 1; all the assets were given equal weight w_k; the time to expiry was 0.1;

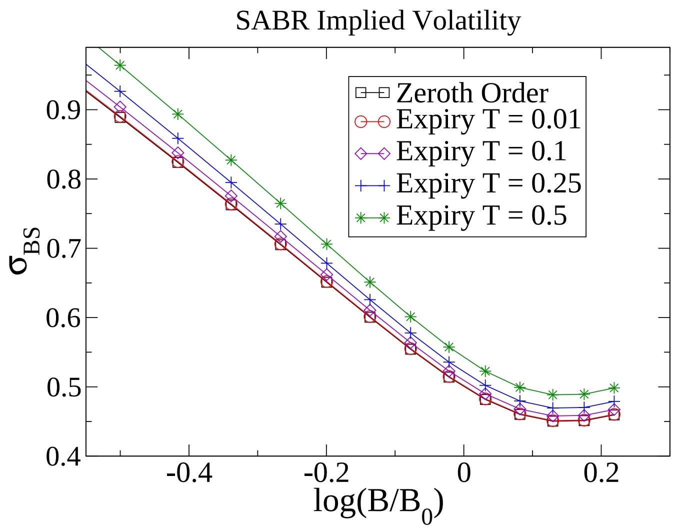

Figure 1. The implied volatility curve for a single asset case comparing the small expiry time (zeroth order) approximation (Eqn. (4)) and the asymptotic expression from [14] with multiple expiry times. The parameters for this example were $\rho = -0.3$, $a^0 = 0.5$, $\nu = 2.5$, $\beta = 1$, and $f^0 = 47$. The zeroth order approximation recovers the Hagan asymptotic expression when the expiry time is set to zero.

Average Solve Time (seconds)

Number of assets	no preconditioner	preconditioner	Monte Carlo	difference
1	< 1	< 1	32	1.2%
10	8	6	744	3.1%
20	68	10	1414	1.7%
50	1180	37	6031	6.8%

Table 1. A comparison of the average solve time required in seconds for the geodesic approach with and without preconditioner and Monte Carlo. The percentage difference between the Monte Carlo result and the geodesic approach result is also given. The Monte Carlo simulation used 20000 samples. The parameters of the basket (a_k^0, f_k^0, ν_k) used in the tests were generated randomly; β was selected to be 1; all the assets were given equal weight w_k; the time to expiry was 0.1; the assets were uncorrelated. Tests were performed on a xeon processor. The geodesic approach, Eqn. (4), is correct in the limit as expiry time goes to zero. See Figure 1.

the assets were uncorrelated. The preconditioned geodesic approach gives a very close result to the Monte Carlo simulation at a fraction of the computational cost.

To estimate the error due to the volatility and time to expiry, several expiry times are explored using Monte Carlo and compared with the geodesic approach for basket examples consisting of 10, 25, and 50 assets with correlation. Table 2 examines the difference between Monte Carlo and the geodesic approach at a particular log-moneyness value as the time to expiry approaches zero. Figures 2, 3, and 4 show the implied volatility for baskets of 10, 25, and 50 assets at several strike values and multiple expiry times comparing Monte Carlo and the geodesic approach. The parameters used for the baskets were generated randomly, with

Assets	$\log(B/B_0)$	Expiry	diff	Monte Carlo	Geodesic	Steepest Descent
10	-0.0443518	0.25	9.1%	0.128137	0.116462	0.119527
		0.1	3.8%	0.121182		
		0.01	1.0%	0.115363		
25	-0.027446	0.25	2.2%	0.0953285	0.0932209	0.0946336
		0.1	1.1%	0.0942288		
		0.05	0.02%	0.0932449		
50	0.0460439	0.25	20.0%	0.117797	0.0942415	0.0931409
		0.1	5.9%	0.10018		
		0.05	2.3%	0.0964529		

Table 2. A comparison of the implied volatility found using the geodesic approach, Eqn. (4), steepest-descent [3], and Monte Carlo for baskets of 10, 25, and 50 assets at multiple expiry times. The percentage difference between the Monte Carlo results and the geodesic approach is also given. In all cases, as the expiry time approaches zero, the Monte Carlo result converges to that found from solving Eqn. (4). While the steepest-descent result in [3] is based on a slightly different stochastic volatility process than SABR, it is trivial to implement and gives remarkable accurate results instantly. Parameters for these tests were generated randomly; the weights of the basket elements were all equal.

$f_k^0 \in [39, 69]$, $a_k^0 \in [0.1, 0.5]$, and $\nu_k \in [0.5, 2.9]$ with $\beta = 1$. For the 10 asset case, the correlation amongst the dW_k terms was in $[-0.25, 0.25]$; for the larger asset cases it was in $[0.0, 0.1]$. The correlation between the dZ_k terms and dW_k terms was zero in all the examples, as was the correlation amongst the dZ_k terms. All assets were given equal weight, w_k.

Comparisons of the implied volatility returned by solving Eqn. (4) and that found by Monte Carlo show they converge as the time to expiry approaches zero. For expiry times of 0.1 they differ by a small percentage.

For baskets consisting of more than 80 assets, we found the limited-memory variable metric algorithm insufficient to produce convergent results, even when using the domain decomposition preconditioner.

4. Conclusion

We have presented a deterministic numerical algorithm to calculate the implied volatility of an option on a basket using a stochastic volatility model with an expression derived and presented previously by M. Avellaneda. The algorithm involves numerically minimizing the geodesic length between the initial and final asset states by varying the final state subject to a constraint. Using the SABR stochastic volatility model, examples of baskets consisting of 10, 25, and 50 assets were examined at several strike prices using both the geodesic approach and Monte Carlo for verification. As the expiry time approached zero, the Monte Carlo results converged to the geodesic approach. The geodesic approach is significantly faster than the Monte Carlo approach.

We have also presented results from a domain decomposition preconditioner which substantially improves the constrained minimization performance. While additive Schwarz type preconditioning is frequently used in solving partial differential equations, we find it also has a substantial impact when used with the geodesic algorithm presented.

Several improvements to the algorithm and preconditioner can yet be made. Incorporating the first order term for the diffusion kernel would improve the implied volatility solution for longer expiry times. Adding overlap communication to the domain decomposition preconditioner could improve the preconditioning results even further and potentially resolve

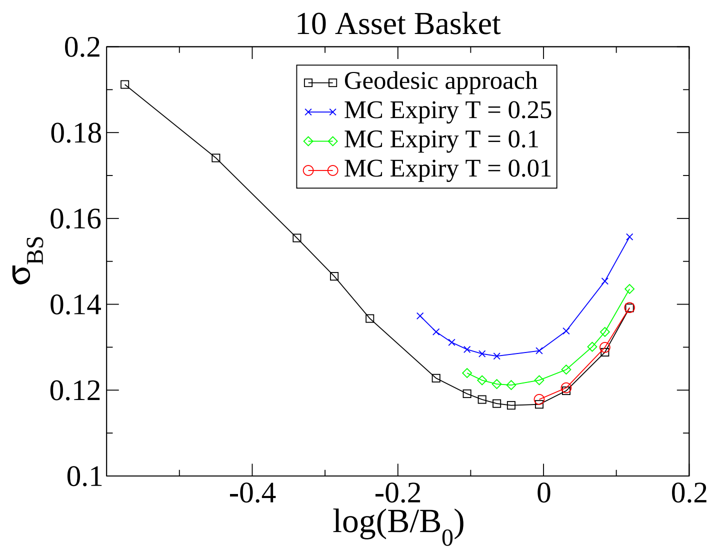

Figure 2. The implied volatility at several strike prices from simulated SABR parameters comparing the geodesic approach with Monte Carlo for a basket of 10 assets with multiple expiry times. The points labeled ”geodesic approach” used Eqn. (4) for the solution. The Monte Carlo simulations required as many as 1×10^8 samples to converge adequately and needed a few hours of run-time on one hundred processors. The geodesic approach required about a minute of run-time on a single processor. As the expiry time approaches zero, the differences between Monte Carlo and Eqn. (4) converge away.

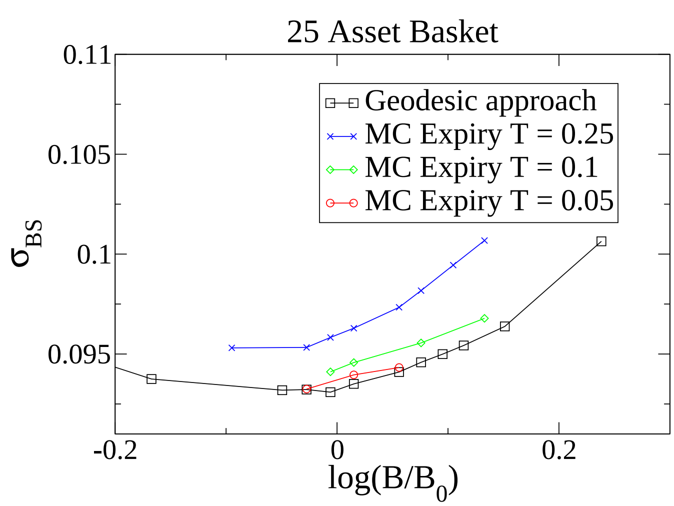

Figure 3. The implied volatility curve produced from simulated SABR parameters comparing the geodesic approach with Monte Carlo for a basket of 25 assets. As the expiry time approaches zero, the Monte Carlo result converges to Eqn. (4).

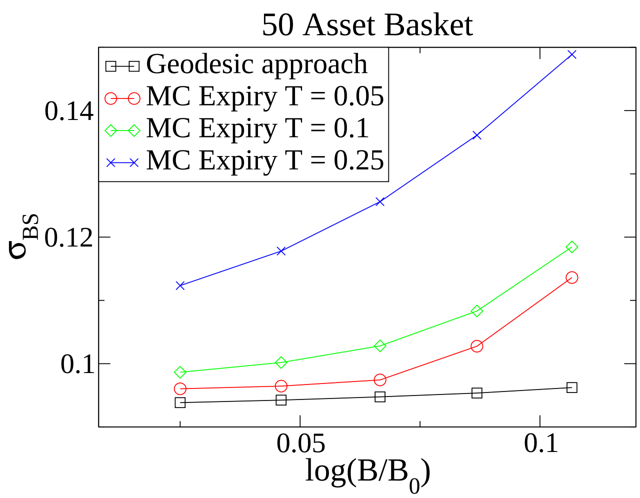

Figure 4. The implied volatility curve produced from simulated SABR parameters comparing the geodesic approach with Monte Carlo for a basket of 50 assets. As the expiry time approaches zero, the Monte Carlo result converges to Eqn. (4). The Monte Carlo simulations used 5×10^5 samples.

the limitations of the optimization algorithm for baskets with more than 80 assets.

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